Space-Filling, Self-Similar Curves of Regular n-Gons

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Definitions ...

Curves in this presentation have **space-filling**, (in some cases) **self-avoiding**, **simple** and **self-similar** properties.
They are curves in the mathematical sense, so they may have corners.

Space-filling – The curve fills an area in the Euclidean plane.
In other words: The points of a line segment can be mapped on an area in the Euclidean plane and vice versa.

Self-avoiding - The curve neither touches nor crosses itself.

Simple - The curve can be drawn in one stroke.

Self-similar - The curve can be dissected into smaller copies of itself.

Space Filling, Self Avoiding, Simple, Self-Similar Curves

Guiseppe Peano, 1890:

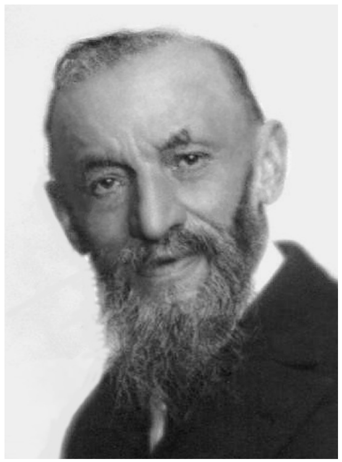
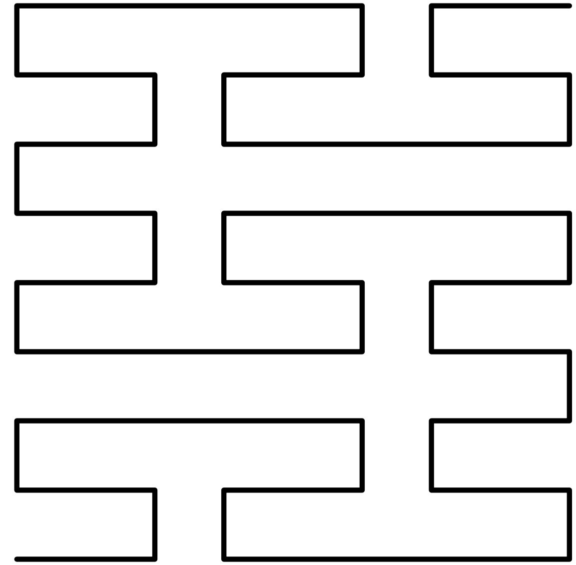
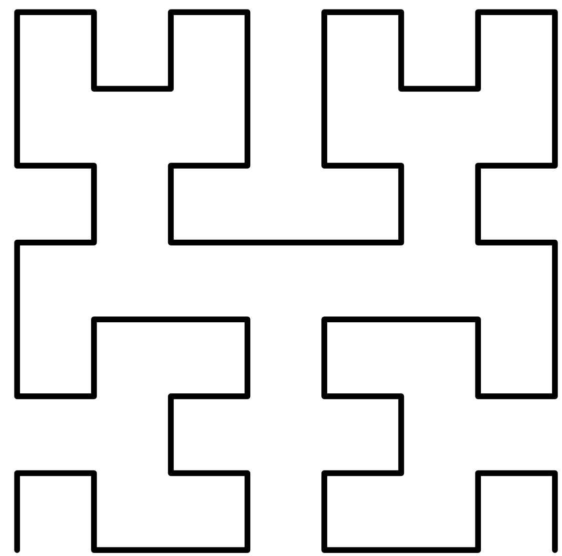


Photo: Author unknown



David Hilbert, 1891:

Photo: Author unknown
possibly by
Constance Reid, 1912



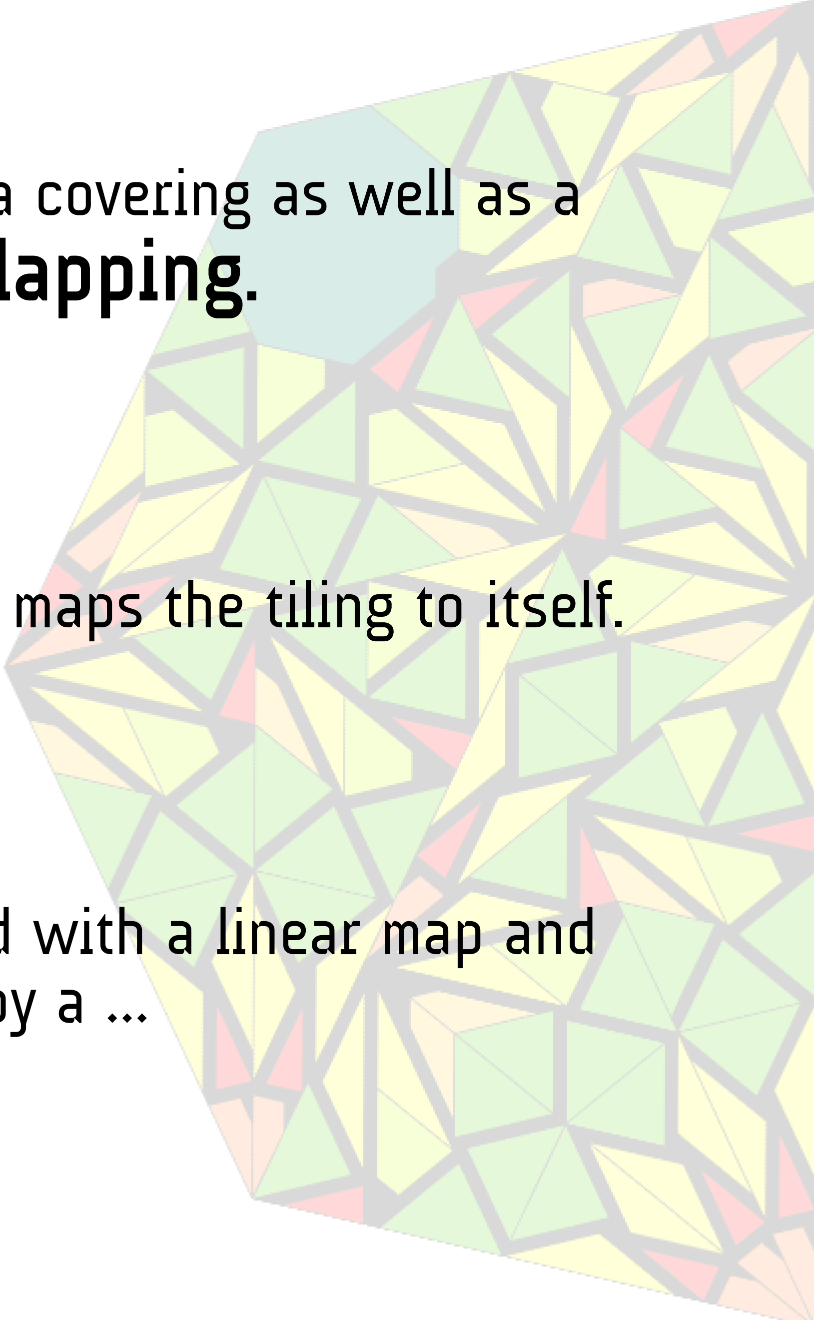
Definitions ...

A **Tiling** is a countable set of tiles, which is a covering as well as a packing of the plane → **No gaps, no overlapping**.

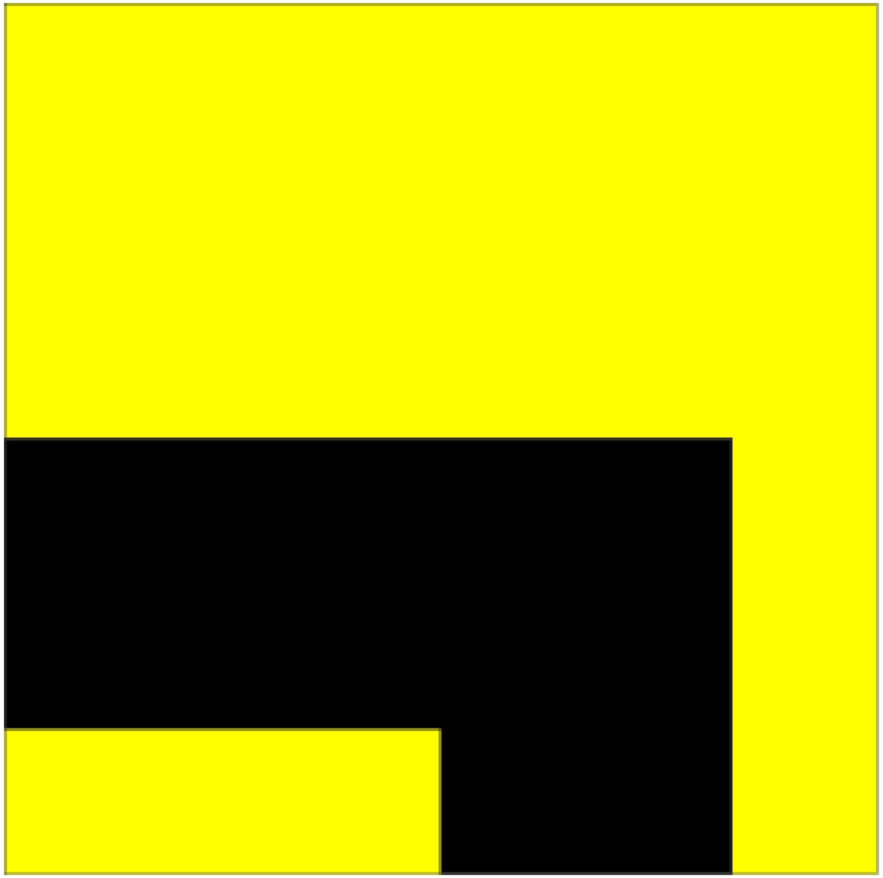
A tiling is called **aperiodic** if no translation maps the tiling to itself.

Substitution means, that a tile is expanded with a linear map and dissected into copies of tiles in original size, by a ...

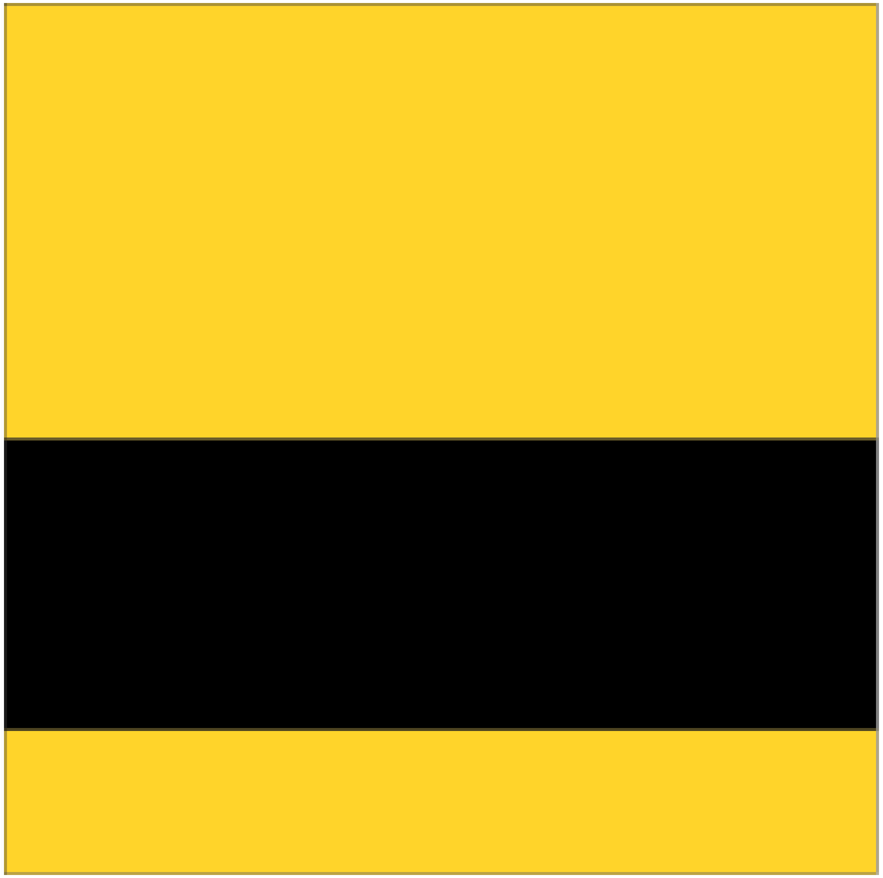
... **Substitution Rule**.



Hilbert Curve and Its Corresponding Aperiodic Tiling



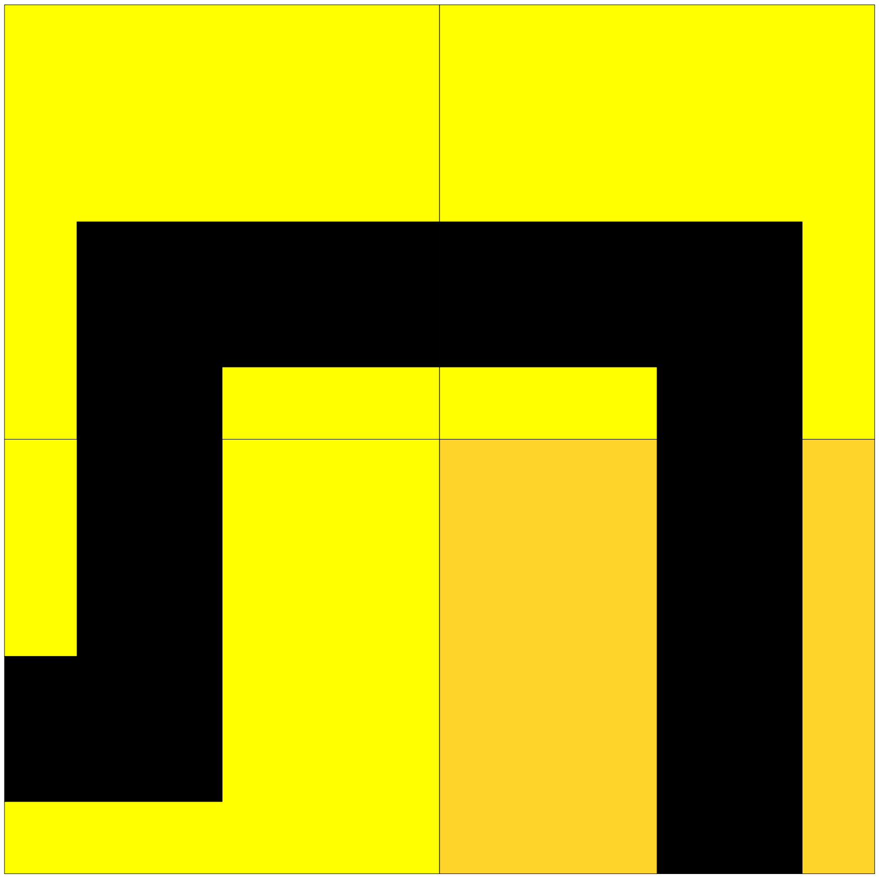
Proto tile 1



Proto tile 2

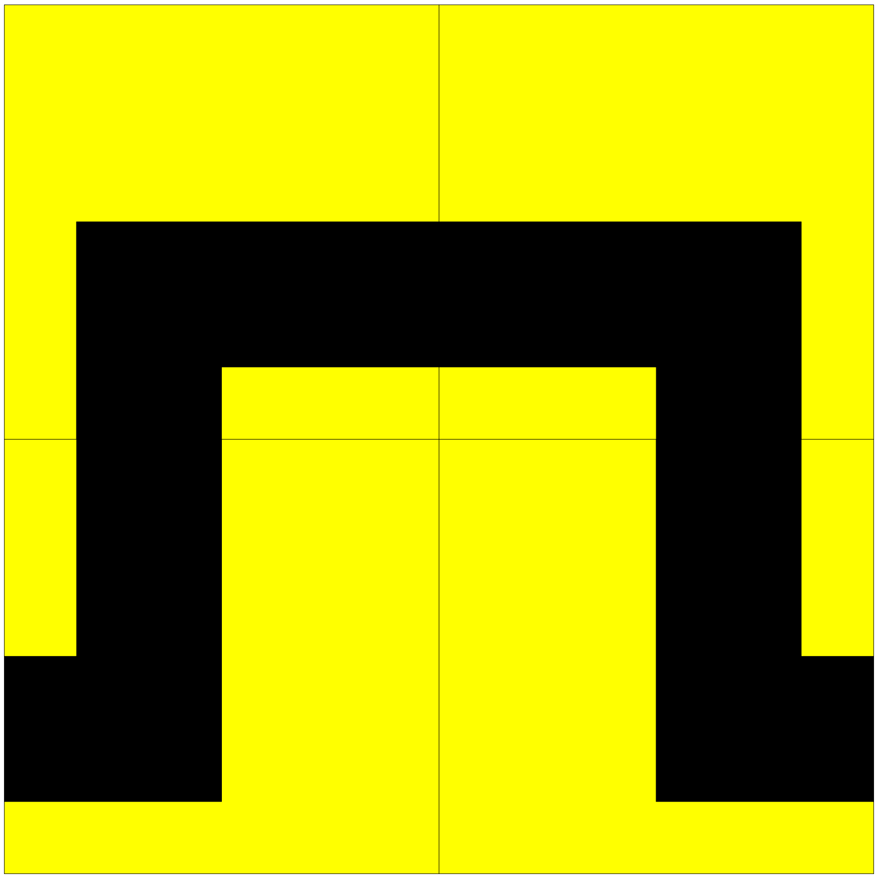
Rotations and reflections are allowed ...

Hilbert Curve and Its Corresponding Aperiodic Tiling



Level-1-supertile 1

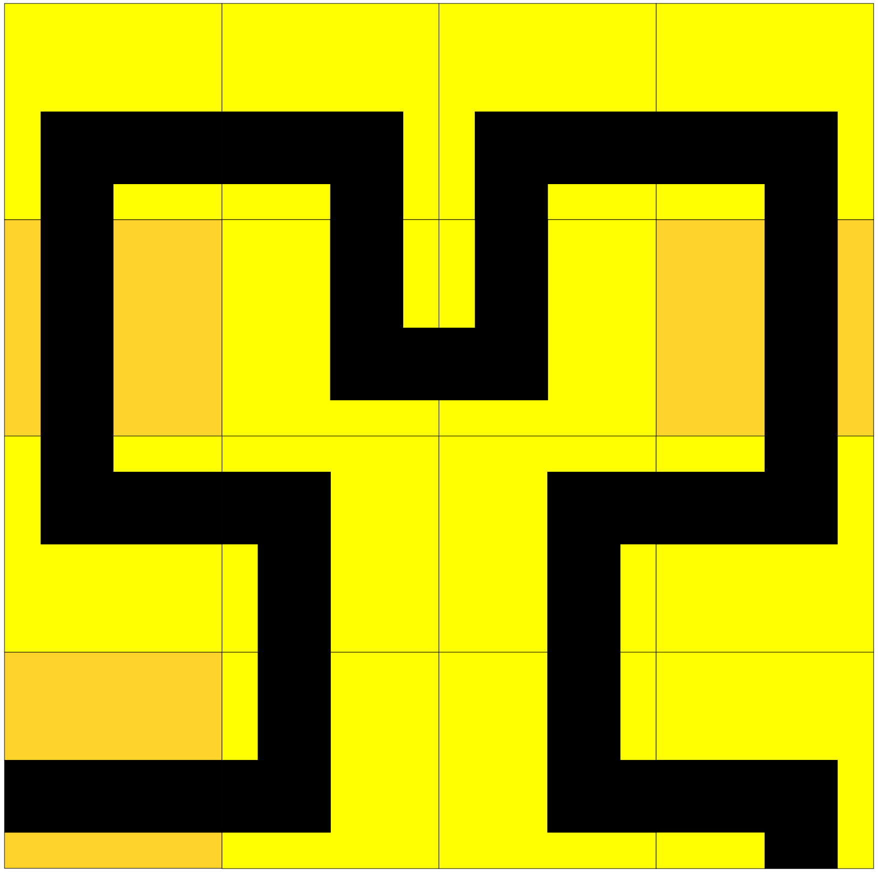
= Substitution rule 1



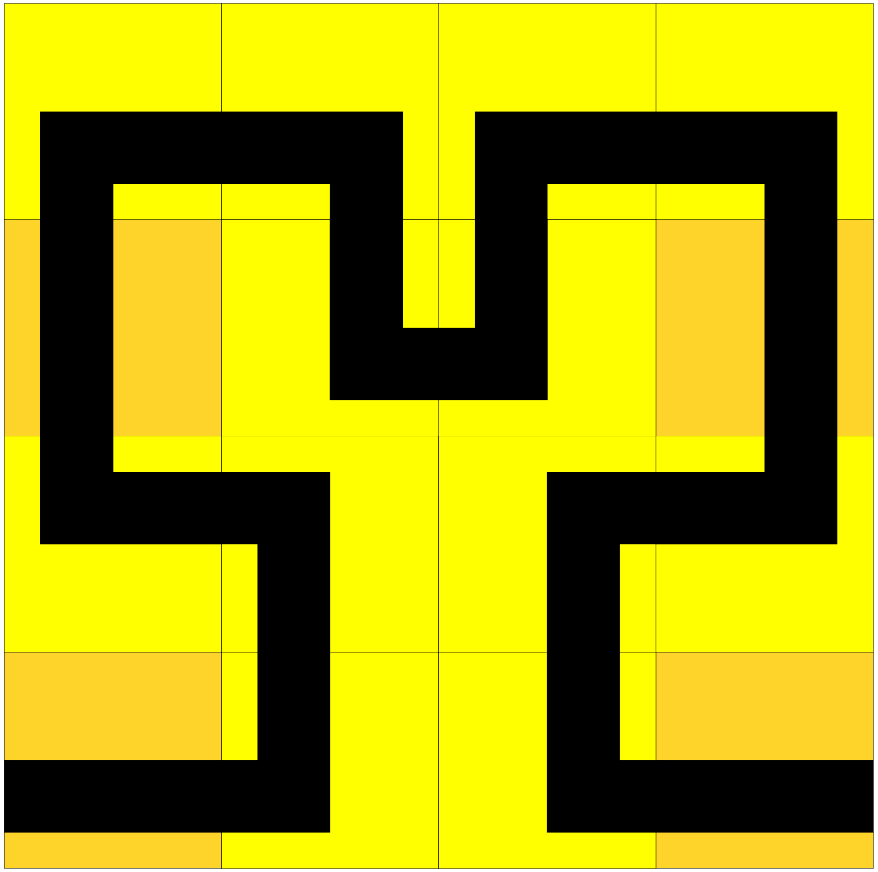
Level-1-supertile 2

= Substitution rule 2

Hilbert Curve and Its Corresponding Aperiodic Tiling

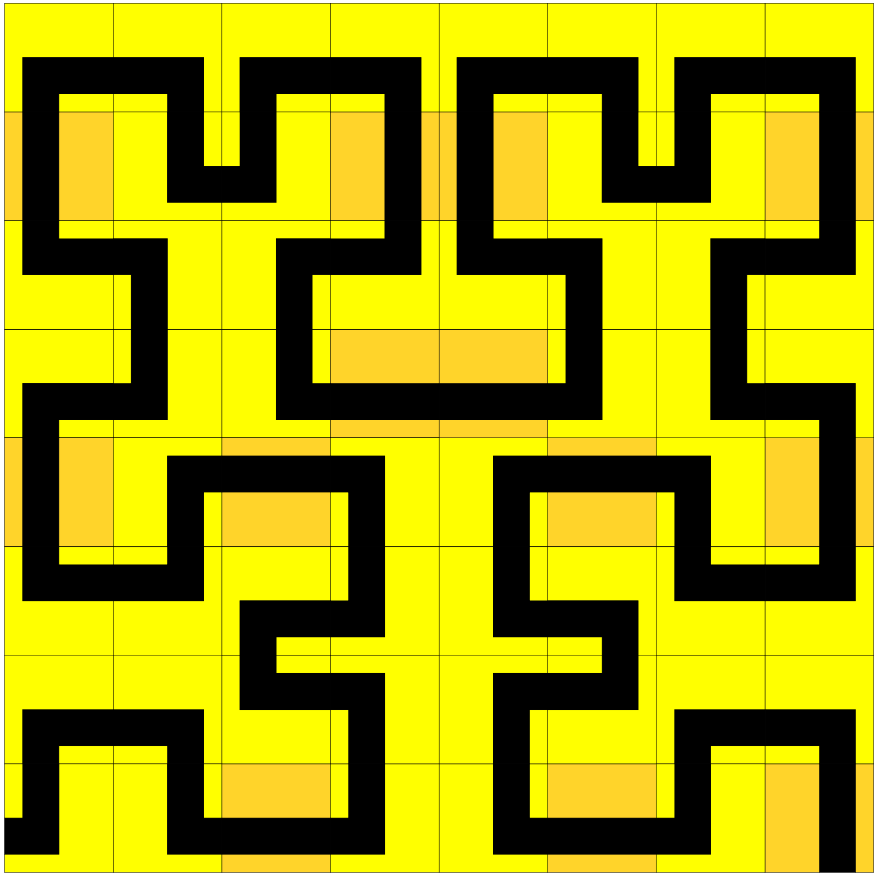


Level-2-supertile 1

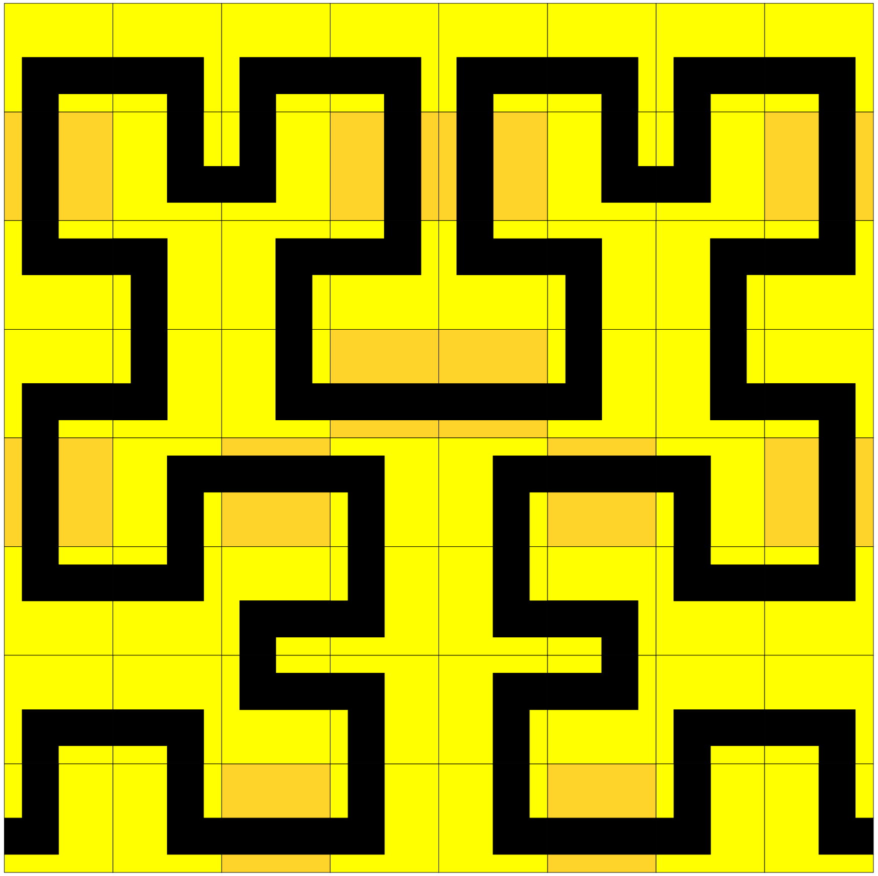


Level-2-supertile 2

Hilbert Curve and Its Corresponding Aperiodic Tiling

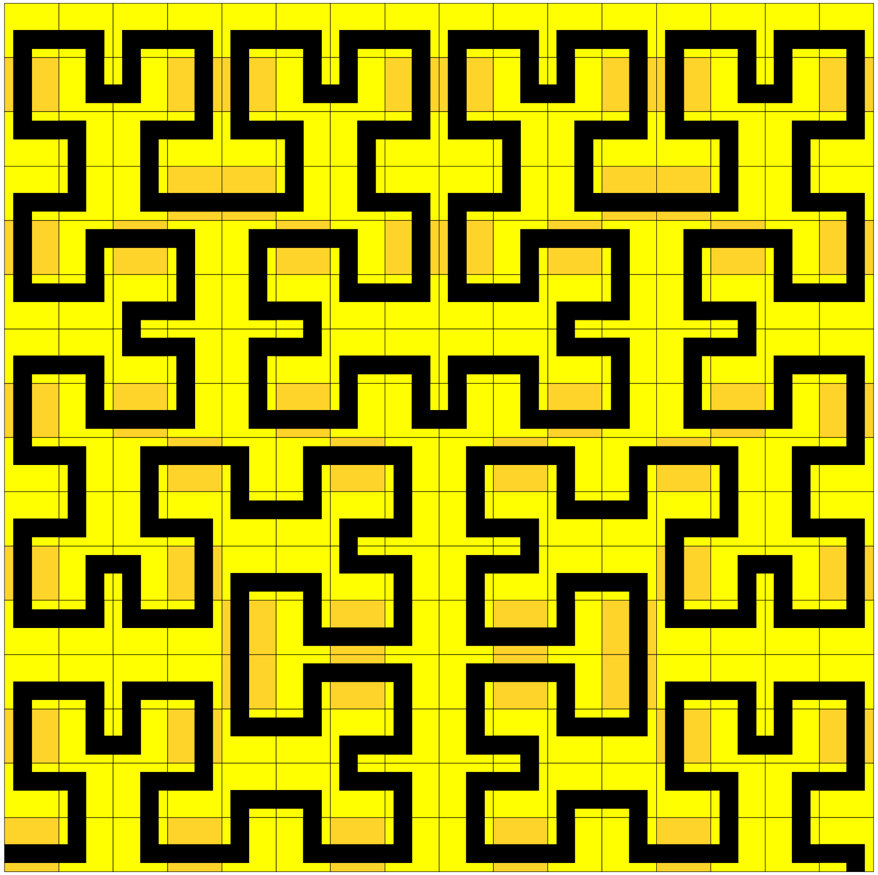


Level-3-supertile 1

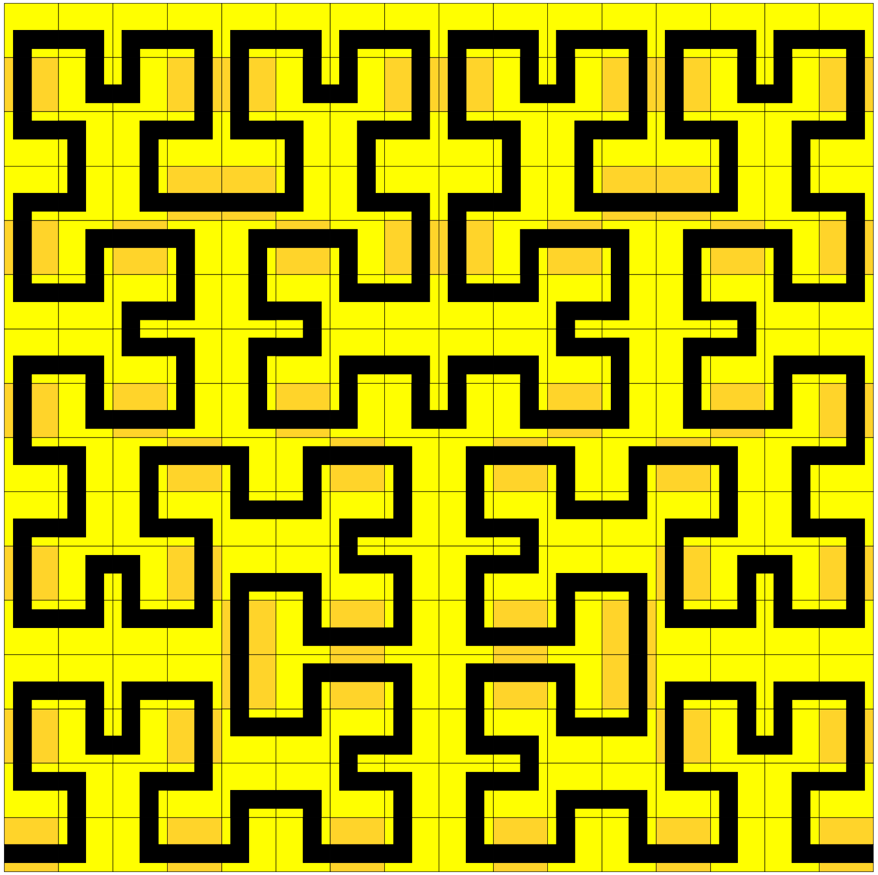


Level-3-supertile 2

Hilbert Curve and Its Corresponding Aperiodic Tiling

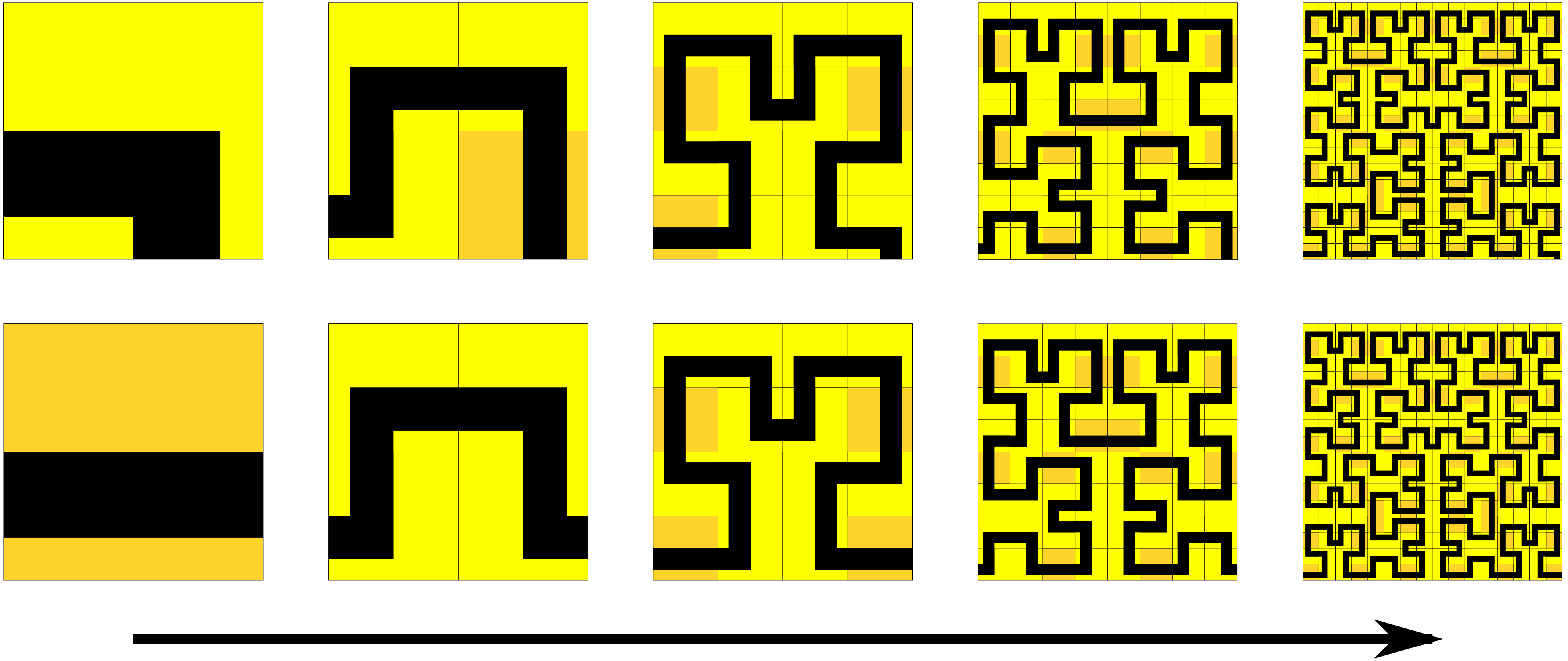


Level-4-supertile 1



Level-4-supertile 2

Hilbert Curve and Its Corresponding Aperiodic Tiling

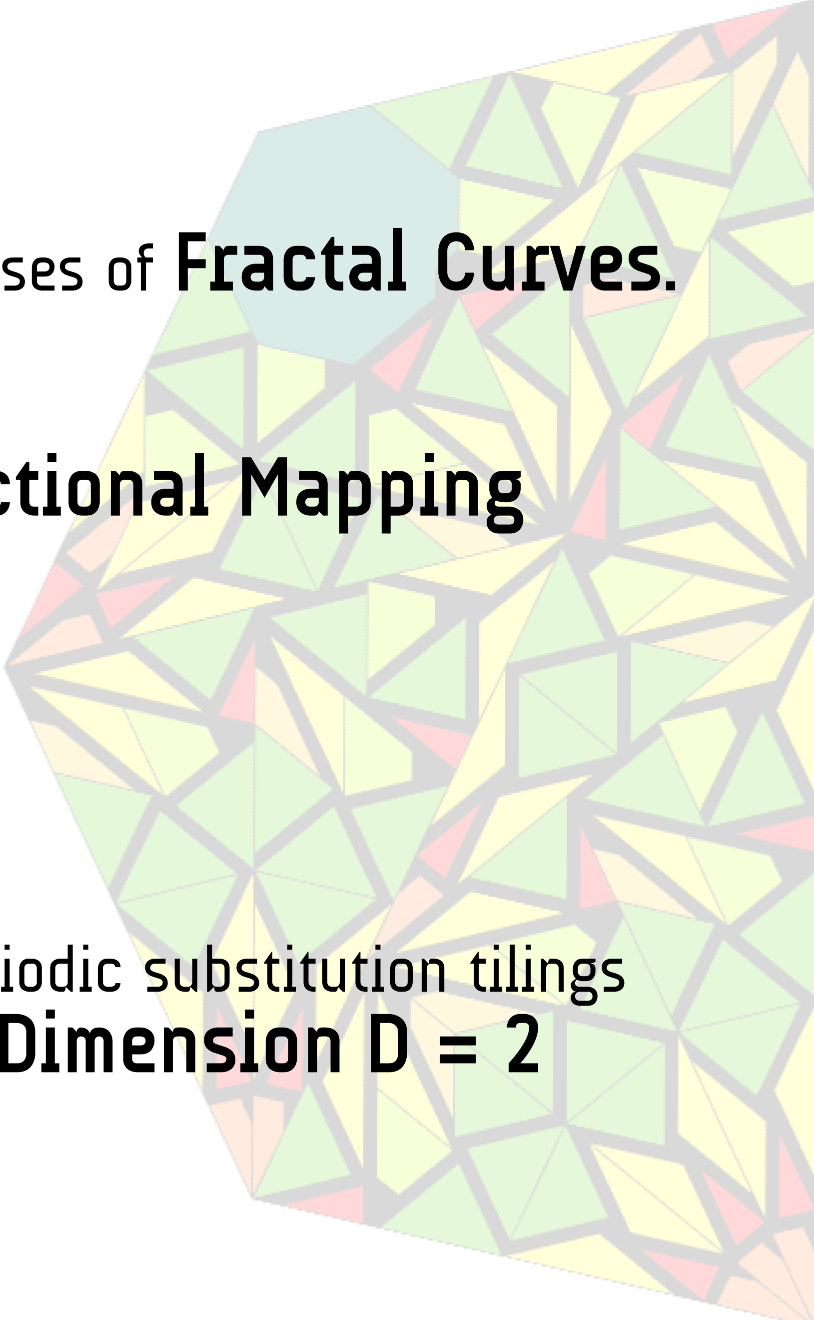


Space filling, self similar curves are special cases of **Fractal Curves.**

In many cases it is possible to find a **Bidirectional Mapping** between ...

- Space filling, self similar curves
- Aperiodic substitution tilings

Both space filling, self similar curves and aperiodic substitution tilings have **Similarity Dimension** or **Fractal Dimension D = 2**



Motivation …

At Bridges 2019 in Linz, Austria the following question was raised at a coffee break with Jörg Arndt, Julia Handl and Jeffrey Ventrella …

Does space-filling, simple, self-similar curves exist, which fill regular n-gons / polygons?

… with n ≥ 5?

We need a ...

Cyclotomic Aperiodic Substitution Tiling (CAST)

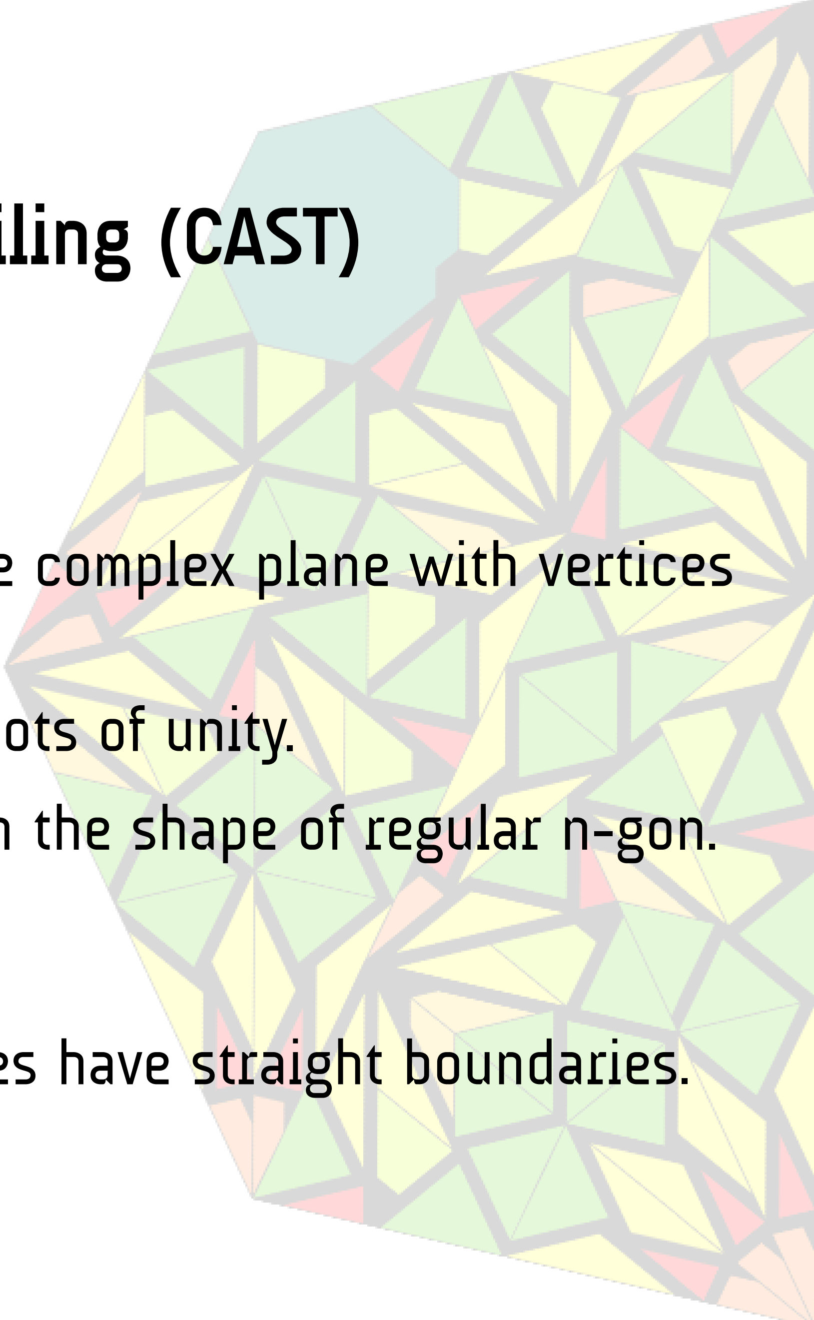
... which is a **Stone Inflation** ...

A **CAST** is an aperiodic substitution tiling in the complex plane with vertices supported by the 2n-th cyclotomic ring $\mathbb{Z}[\zeta_{2n}]$.

... all coordinates can be written as a sum of roots of unity.

... as a result such CAST yield patches or tiles in the shape of regular n-gon.

Stone Inflation means, the level-n-supertiles have straight boundaries.



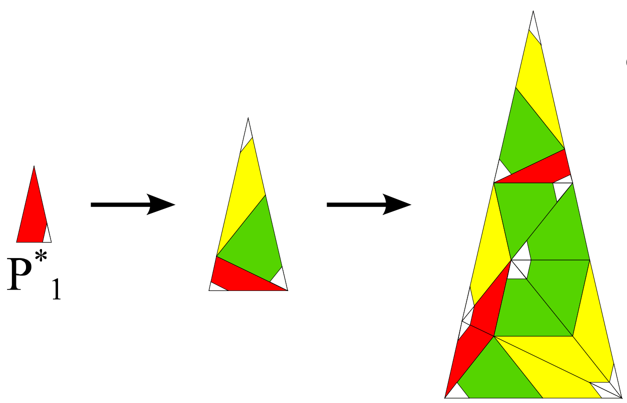
A Cooking Recipe … to Fill the Regular Heptagon

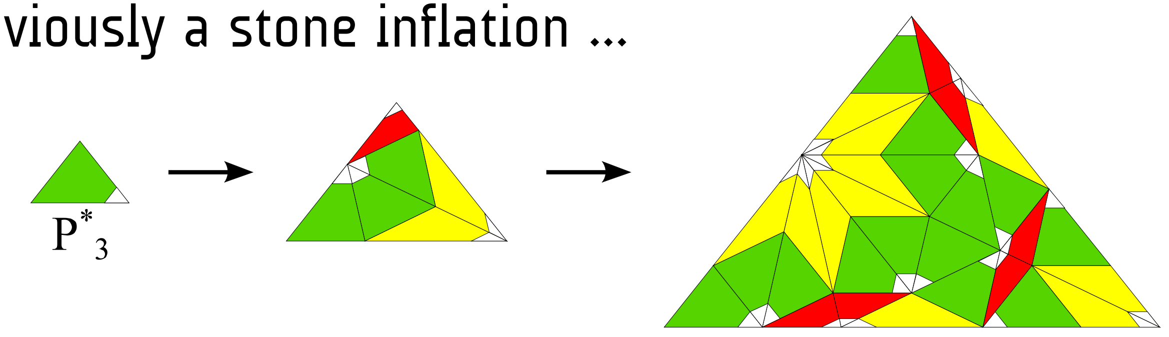
CAST supported by the 14-th cyclotomic ring

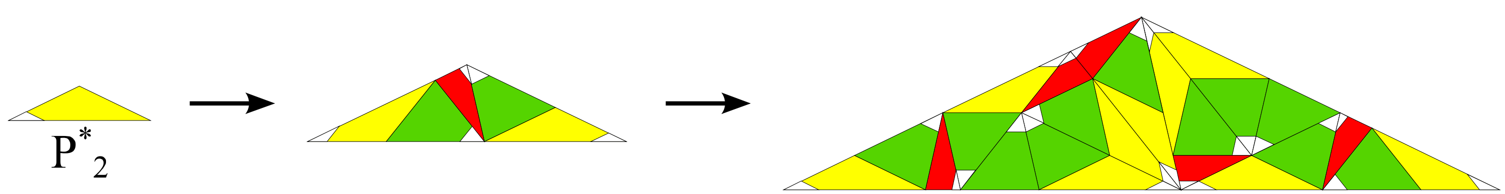
”Danzer's 7-fold variant” in …
Frettlöh, D.; Gähler, F.; Harris, E.O. Tilings Encyclopedia. Available online: http://tilings.math.uni-bielefeld.de/

S. Pautze. “Cyclotomic Aperiodic Substitution Tilings." Symmetry, vol. 9, no. 2:19, 2017, pp. 1–41.

- 3 proto tiles
- All proto tiles in the shape of isosceles triangles
- All isosceles have unit length
- All inner angles are multiples of π/7
- Obviously a stone inflation …







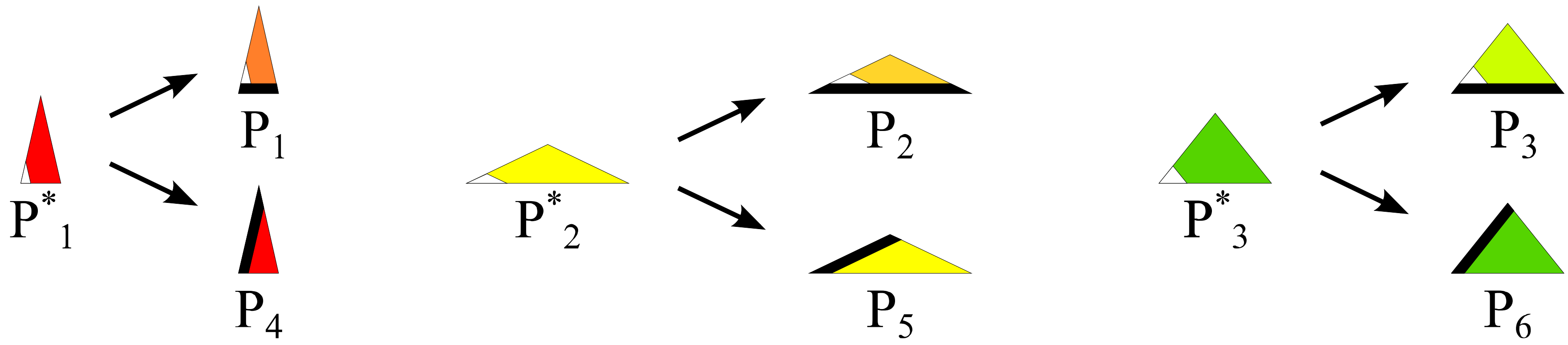
A Cooking Recipe ... to Fill the Regular Heptagon

For every triangle we need to add a “decoration”:

- A line which connects two corner points.

Two version are necessary:

- Base to Base
- Top to Base (top to the other base is also covered, because reflections are allowed)



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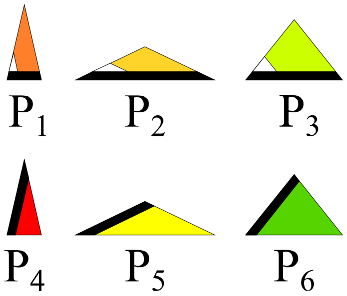
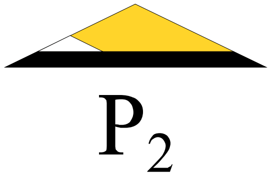


We need to solve six puzzles, one for each substitution rule.

Two corner points have to be connected with proto tiles.

The polygonal chain may touch itself and the sides of the triangle without black line in singular points only.

Example P_2 with heptagonal patch:



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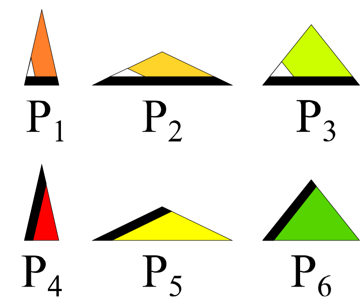

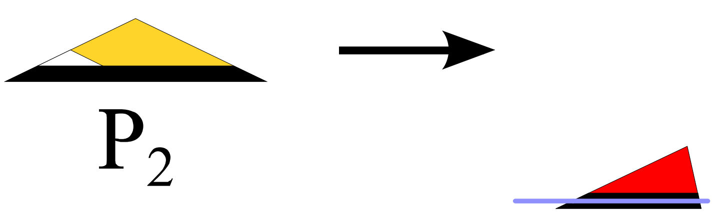

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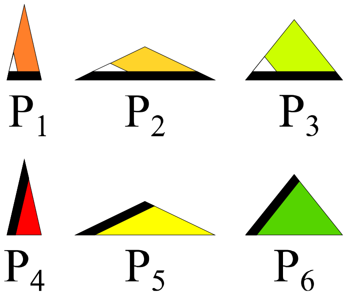

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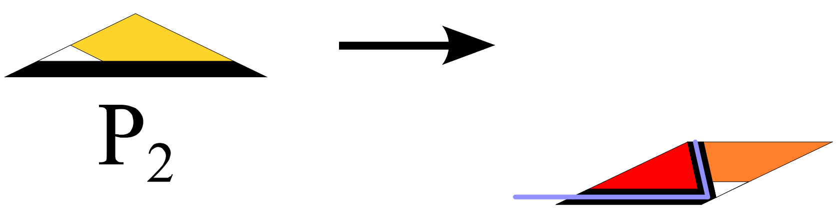

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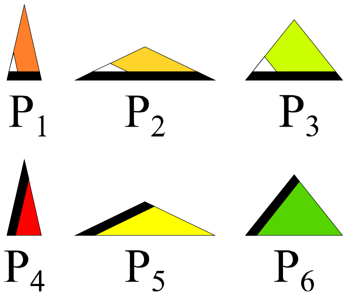

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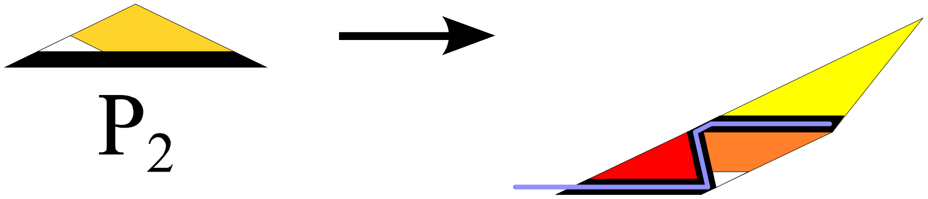

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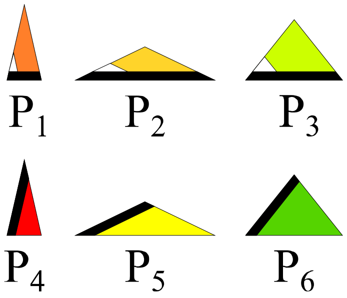

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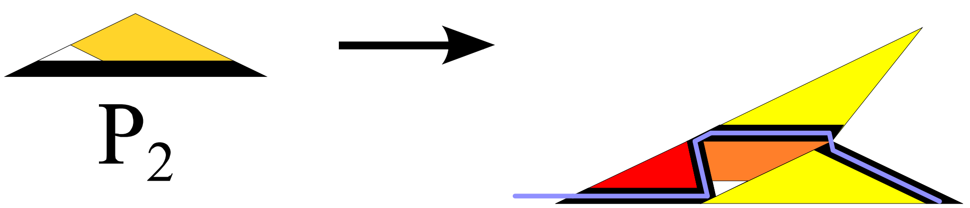

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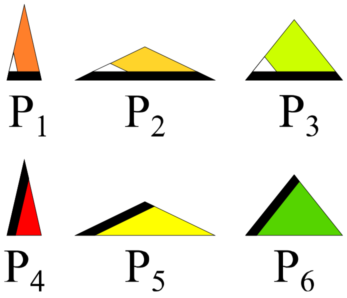

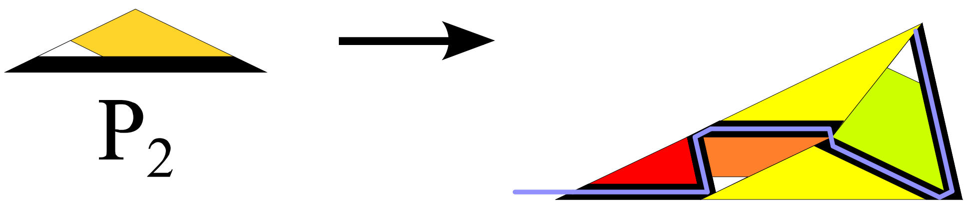

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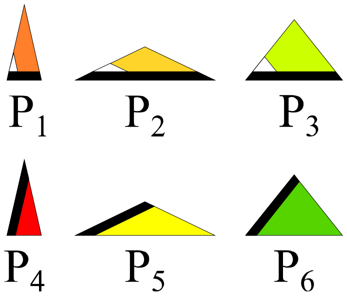

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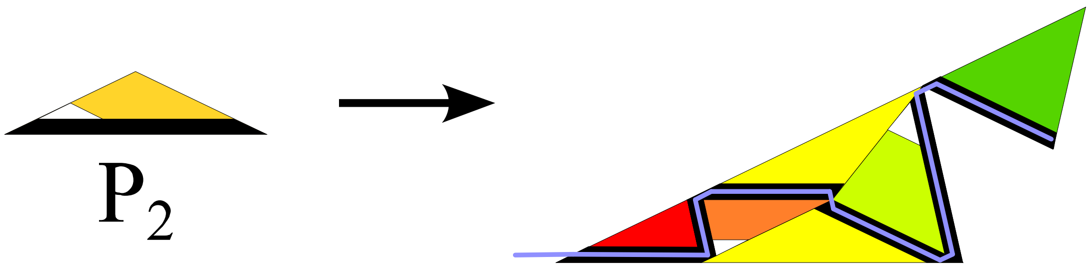

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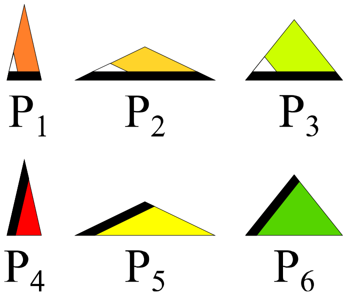

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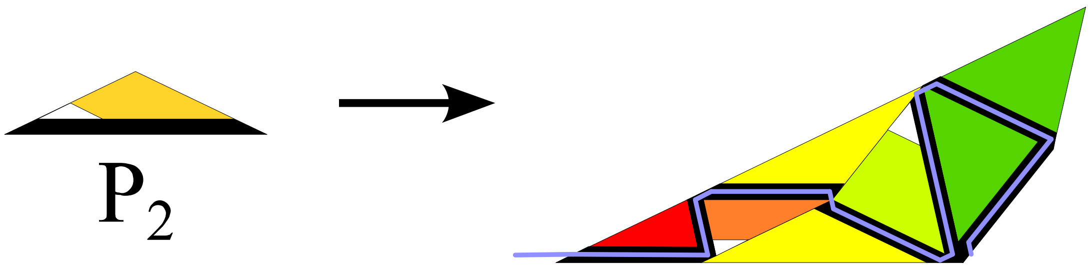

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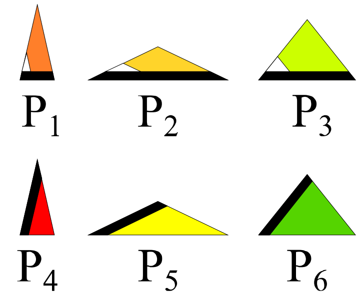

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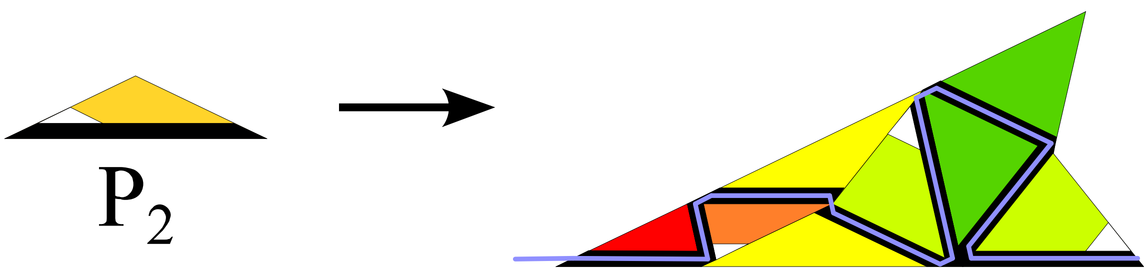

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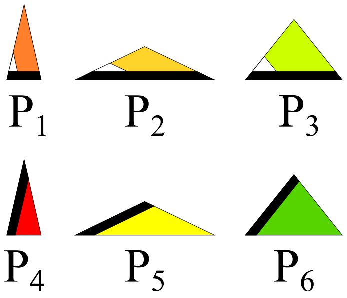

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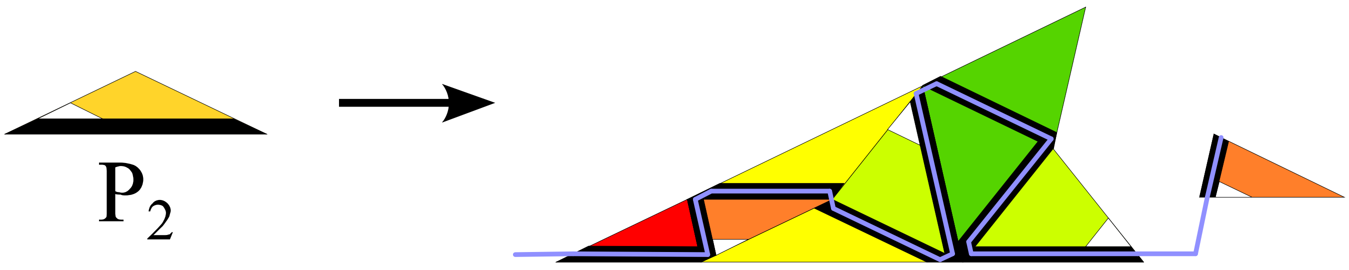

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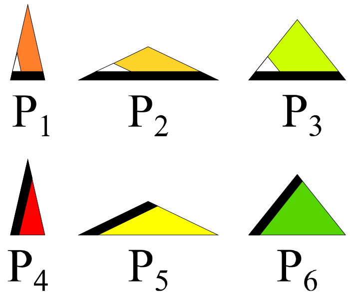

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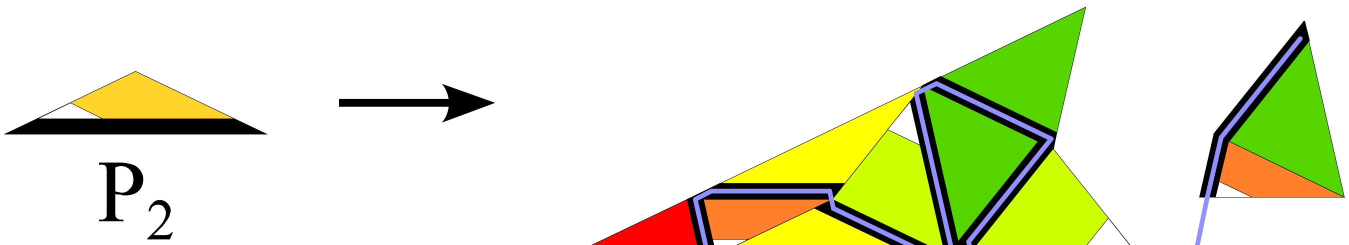

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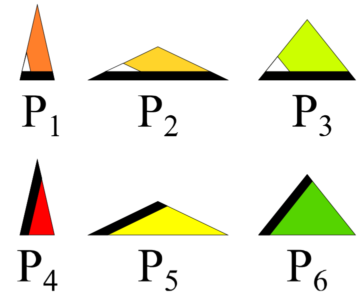

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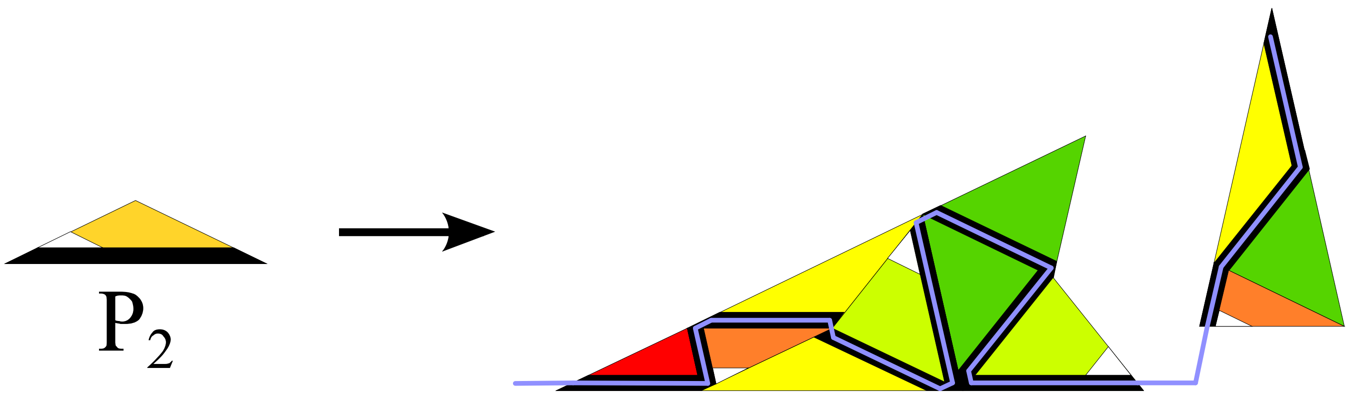

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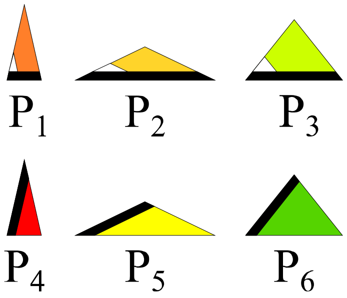

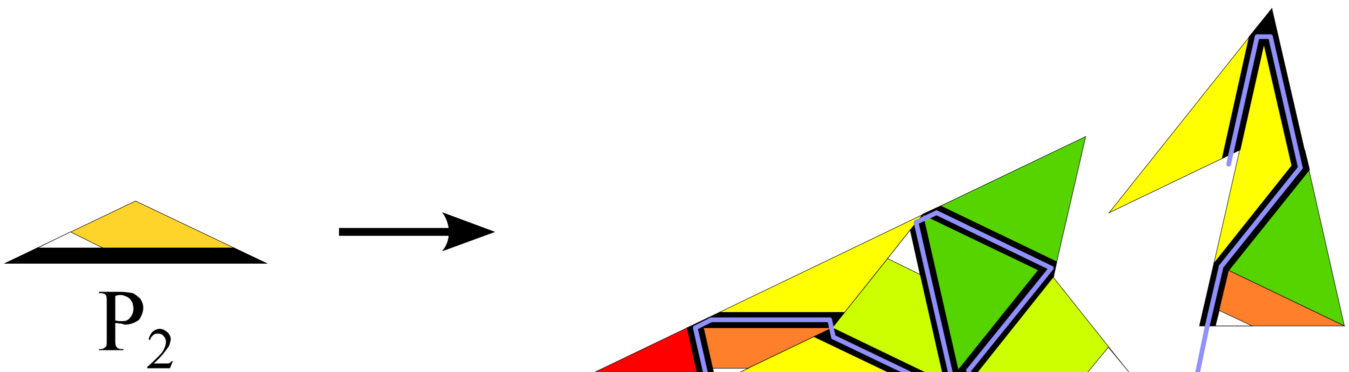

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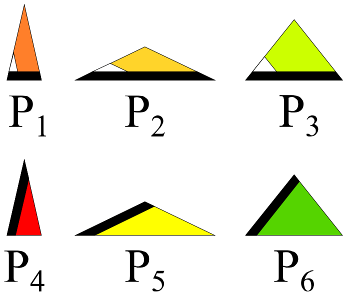

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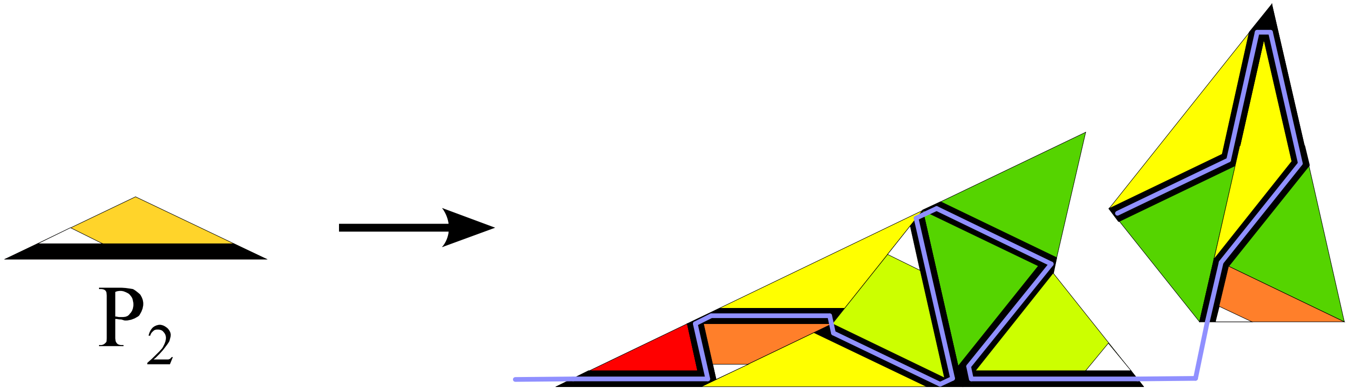

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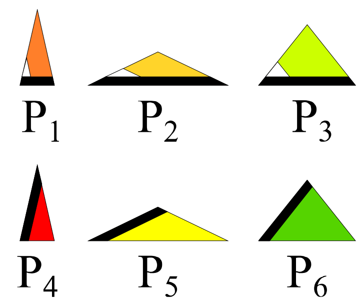

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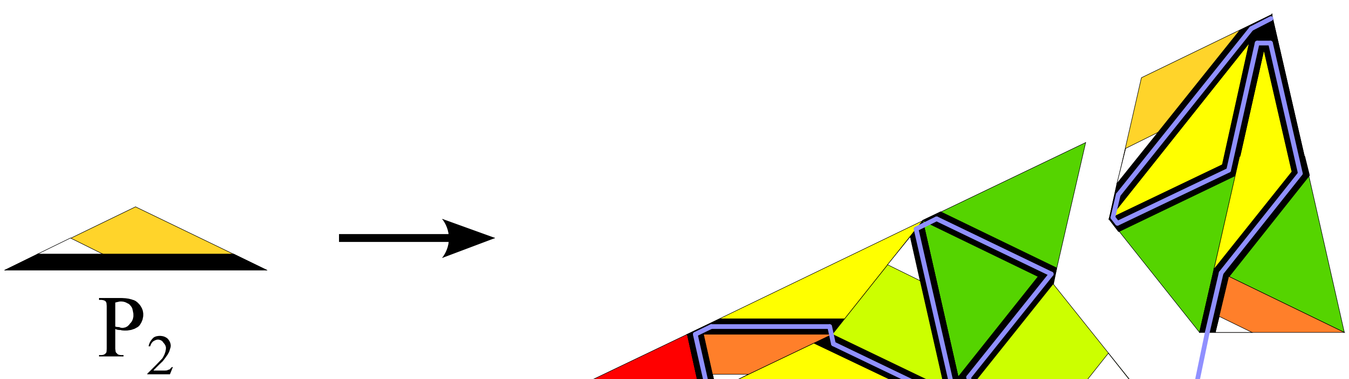

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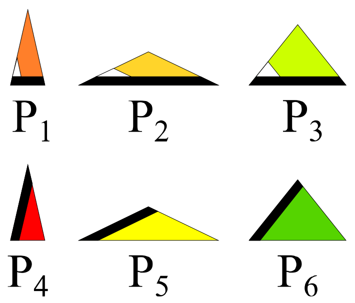

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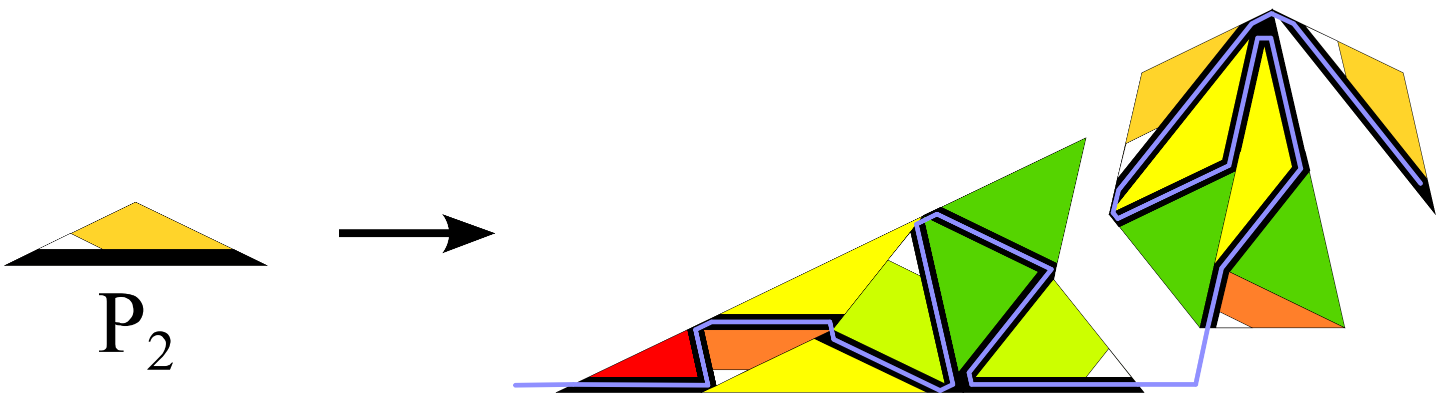

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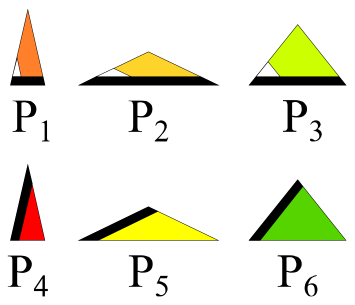

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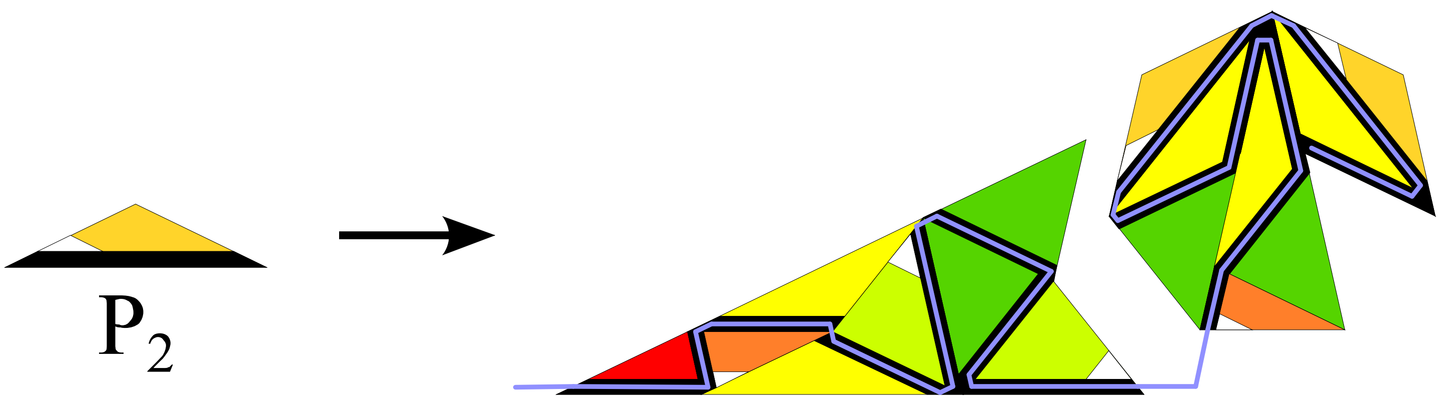

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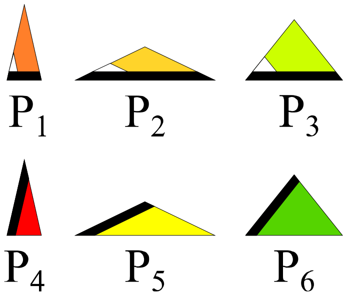
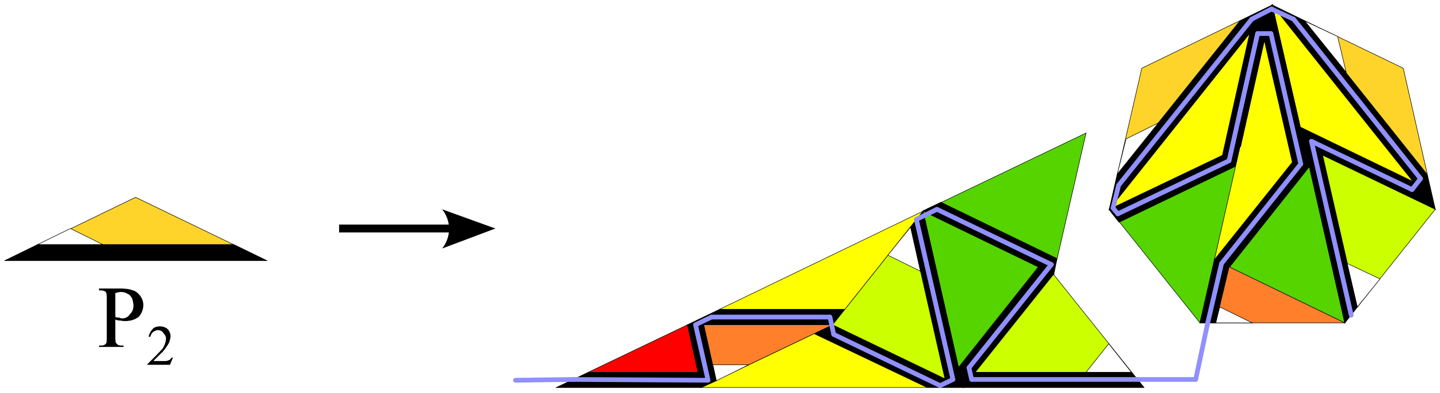
We need to solve six puzzles, one for each substitution rule.

Two corner points have to be connected with proto tiles.

The polygonal chain may touch itself and the sides of the triangle without black line in singular points only.



Example P_2 with heptagonal patch:



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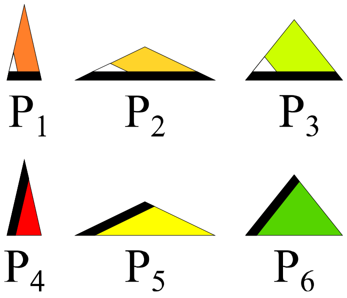

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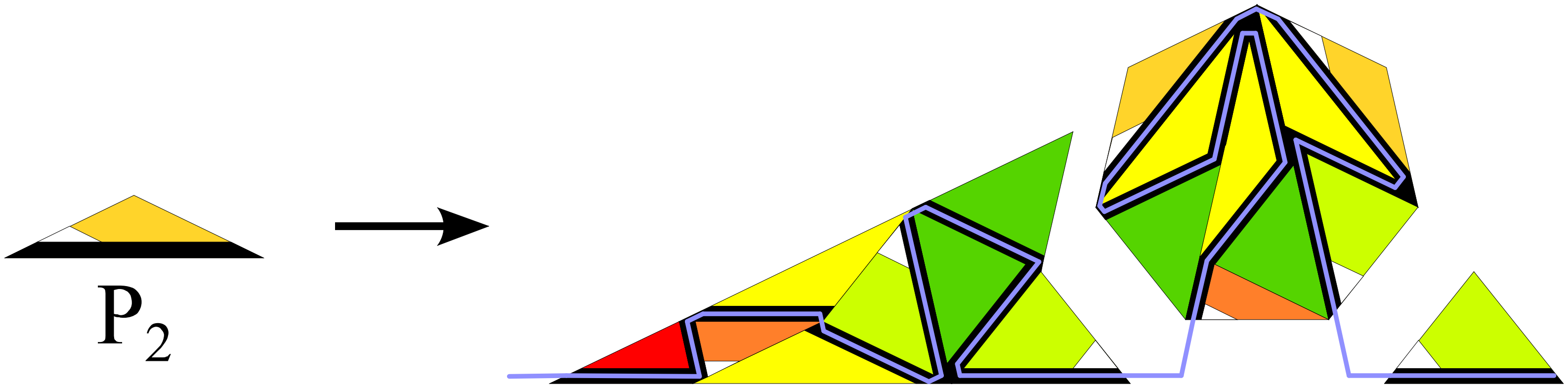

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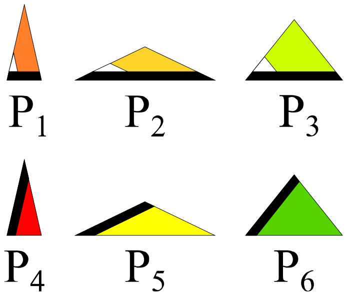

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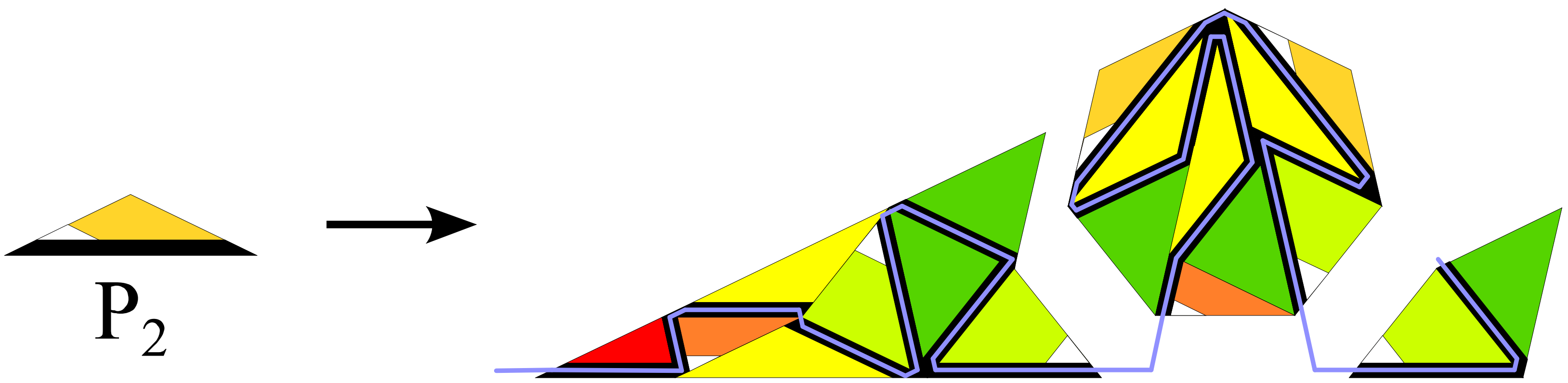

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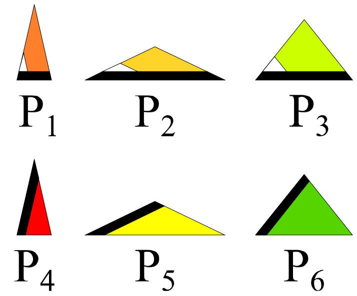

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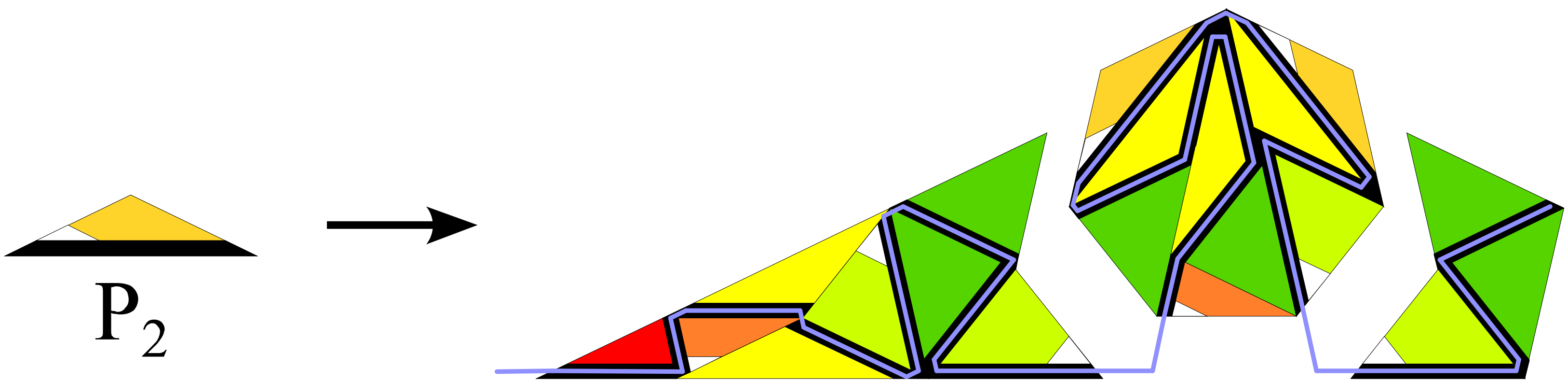

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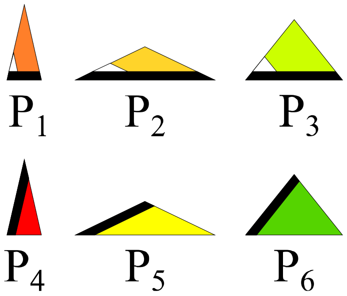

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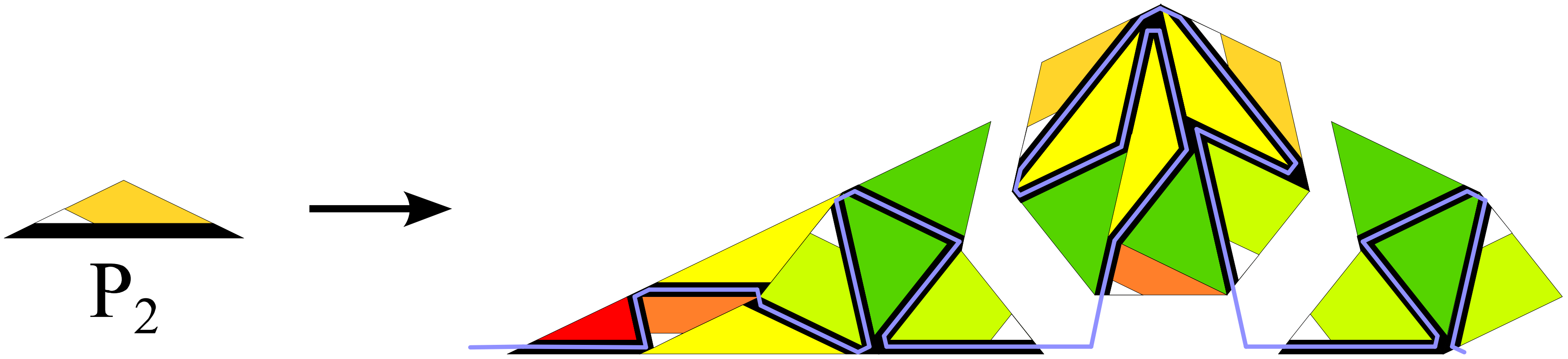

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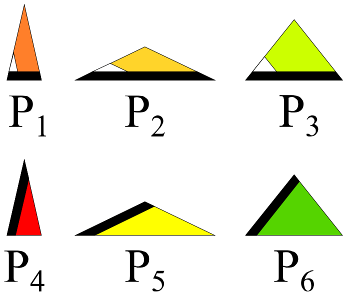

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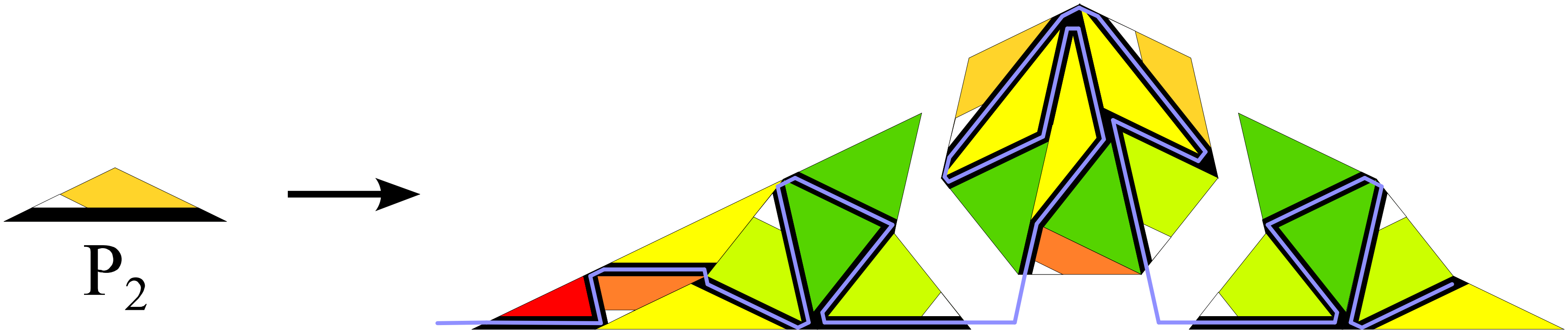

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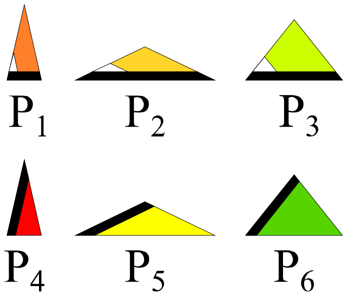

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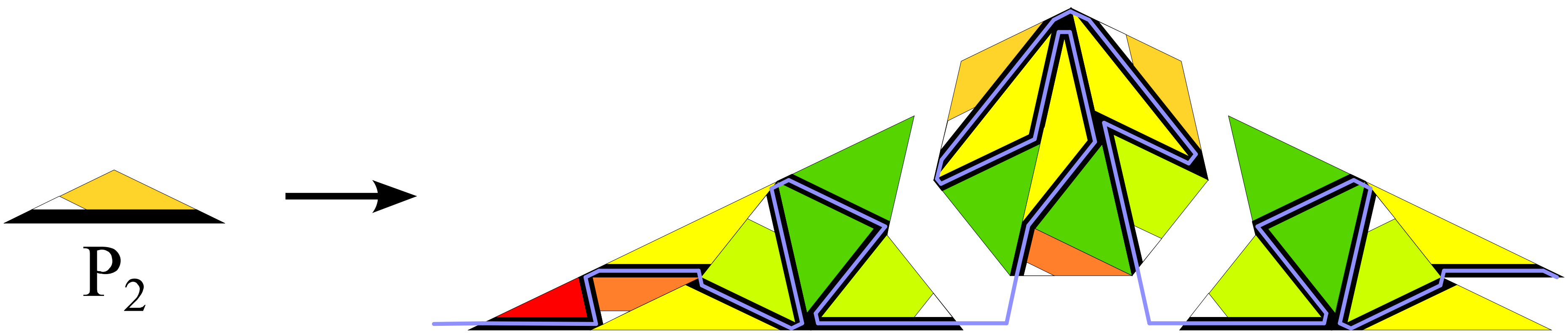

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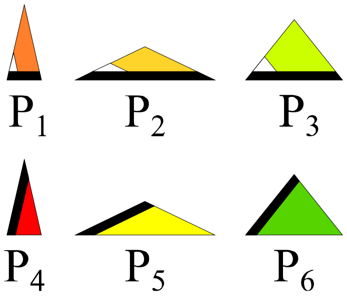

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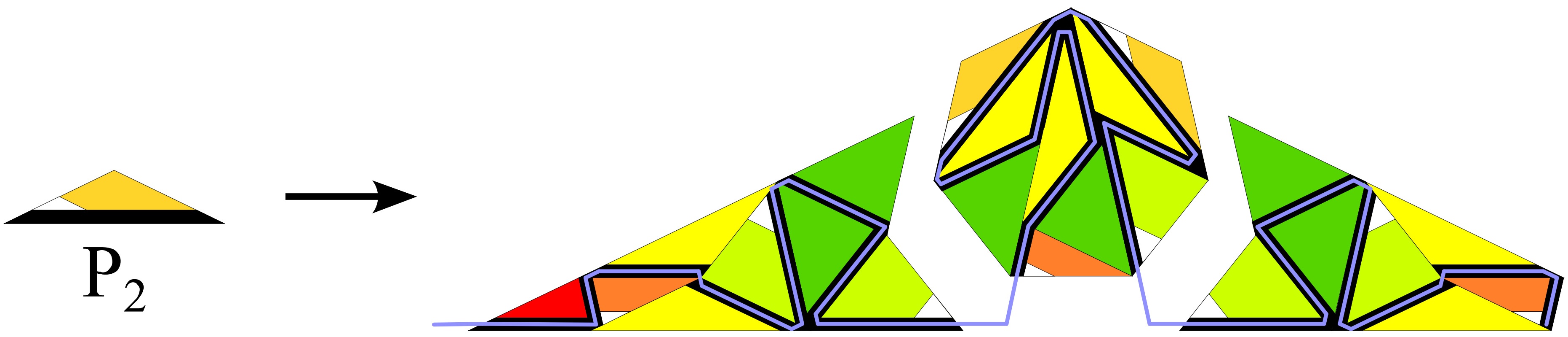

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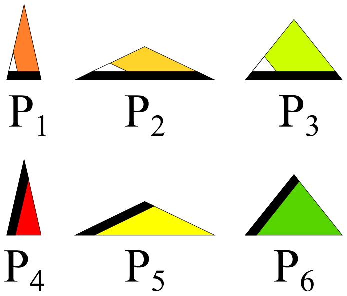

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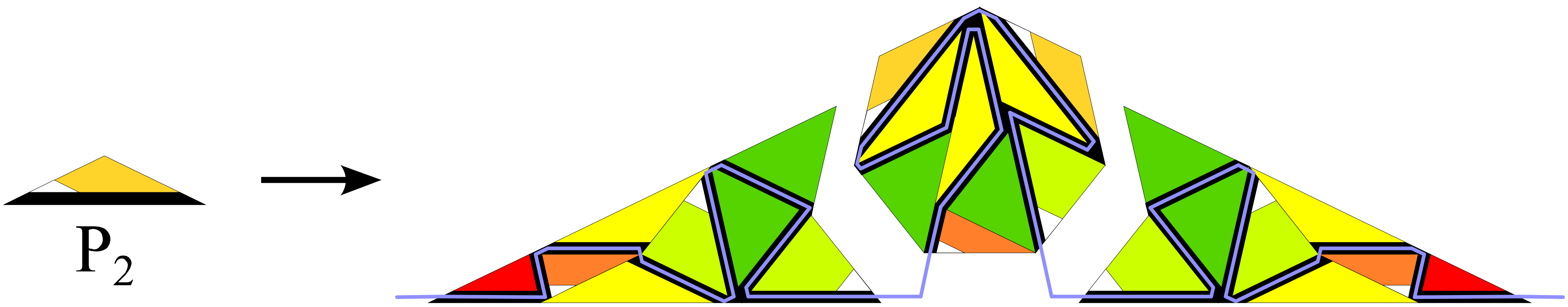
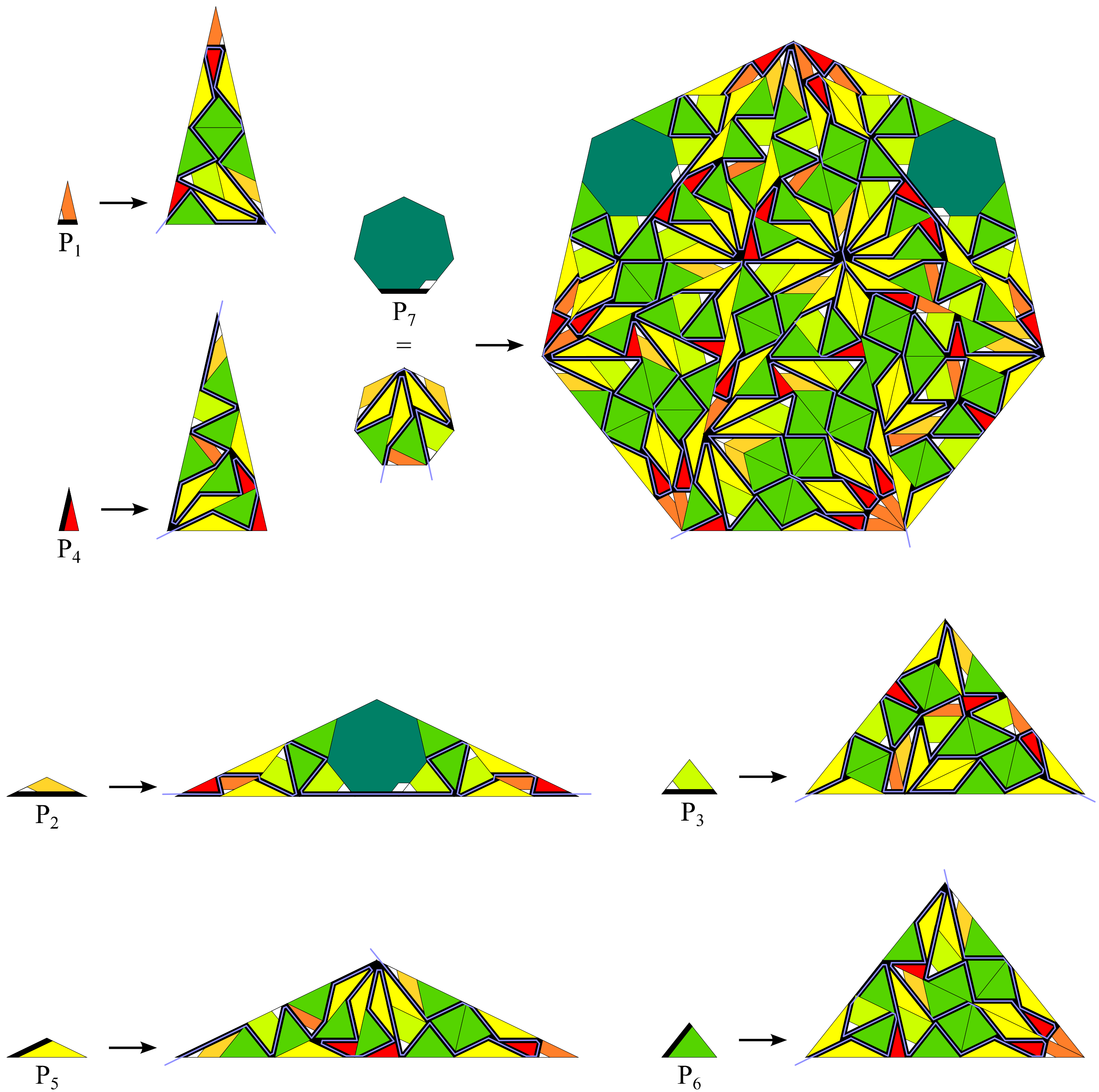


Space Filling, Self Similar Curve of the Regular Heptagon



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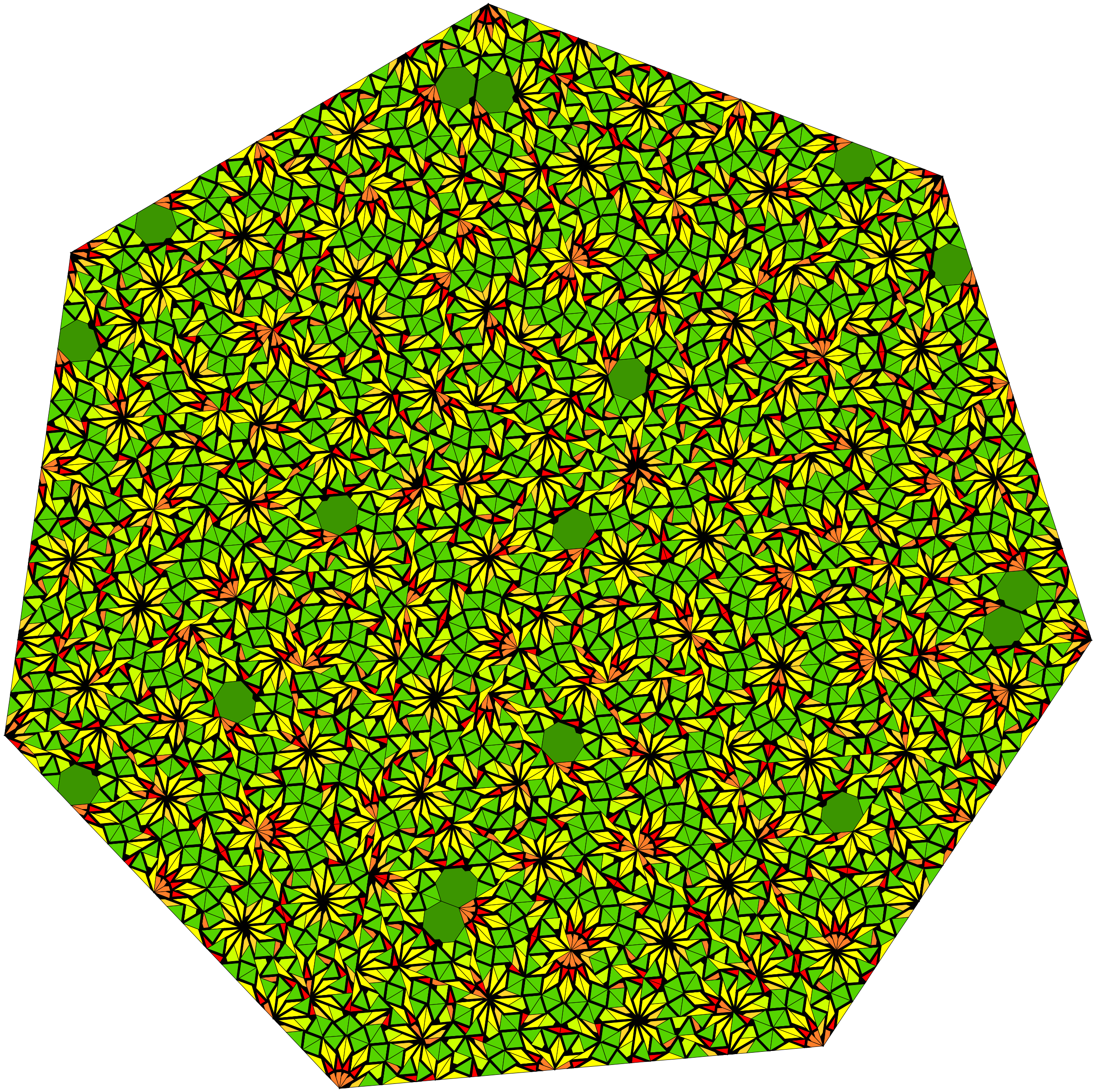
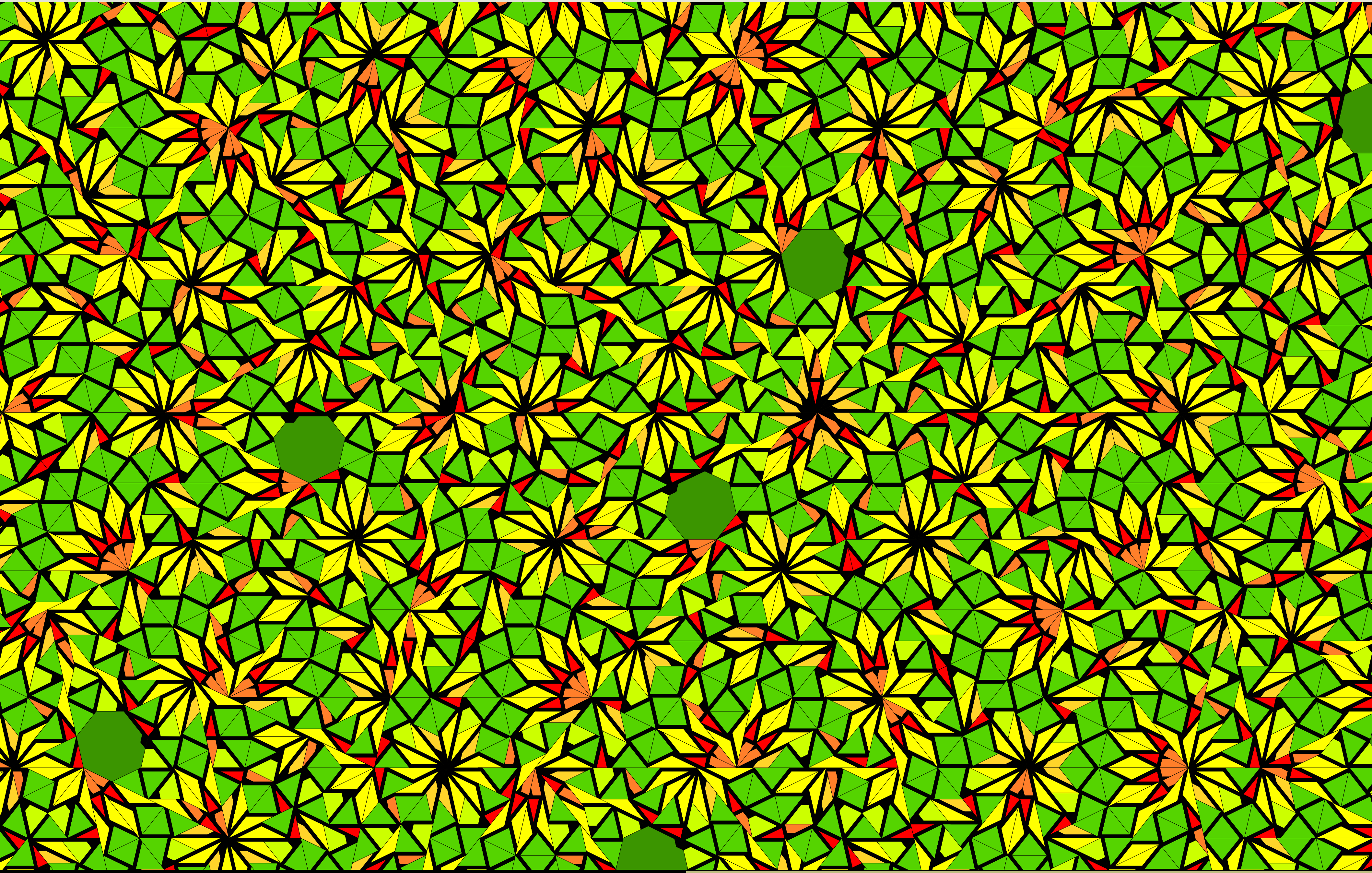


2nd iteration

Space Filling, Self Similar Curve of the Regular Heptagon



Open Questions and Issues …

- The paper describes just a proof of concept.
  There is no proof available (yet) that the algorithm always work.
- With larger n the complexity and the effort raises non-proportional.
- It is very likely that the algorithm works for all $n \geq 7$.
- For $6 \geq n \geq 3$ the algorithm had to be adjusted by shifting the nodes (which connect two neighbour tiles) away from the corner points.

  Surprise!
  As a result of the adjustment the curves yield self avoidance.
- While for $6 \geq n \geq 3$ it is relatively easy to find curves which are self-avoiding it seems to be difficult for $n \geq 7$.

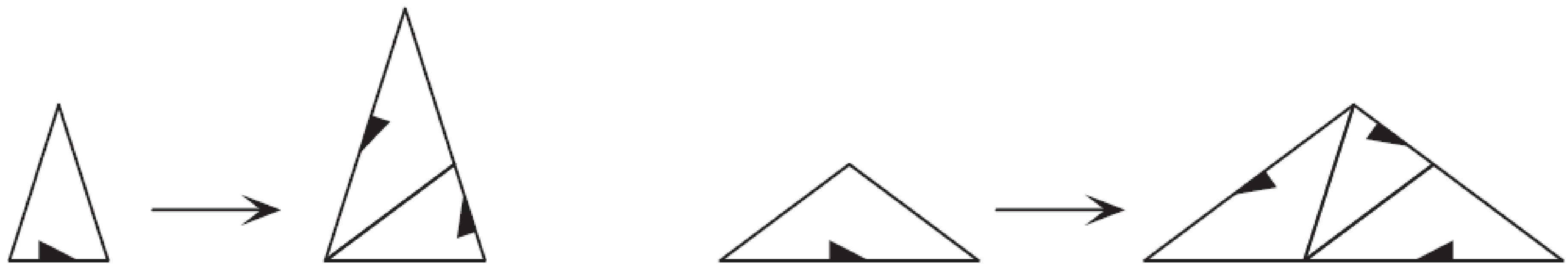
The FASS-Curve of the Regular Pentagon …

Robinson's stone inflation

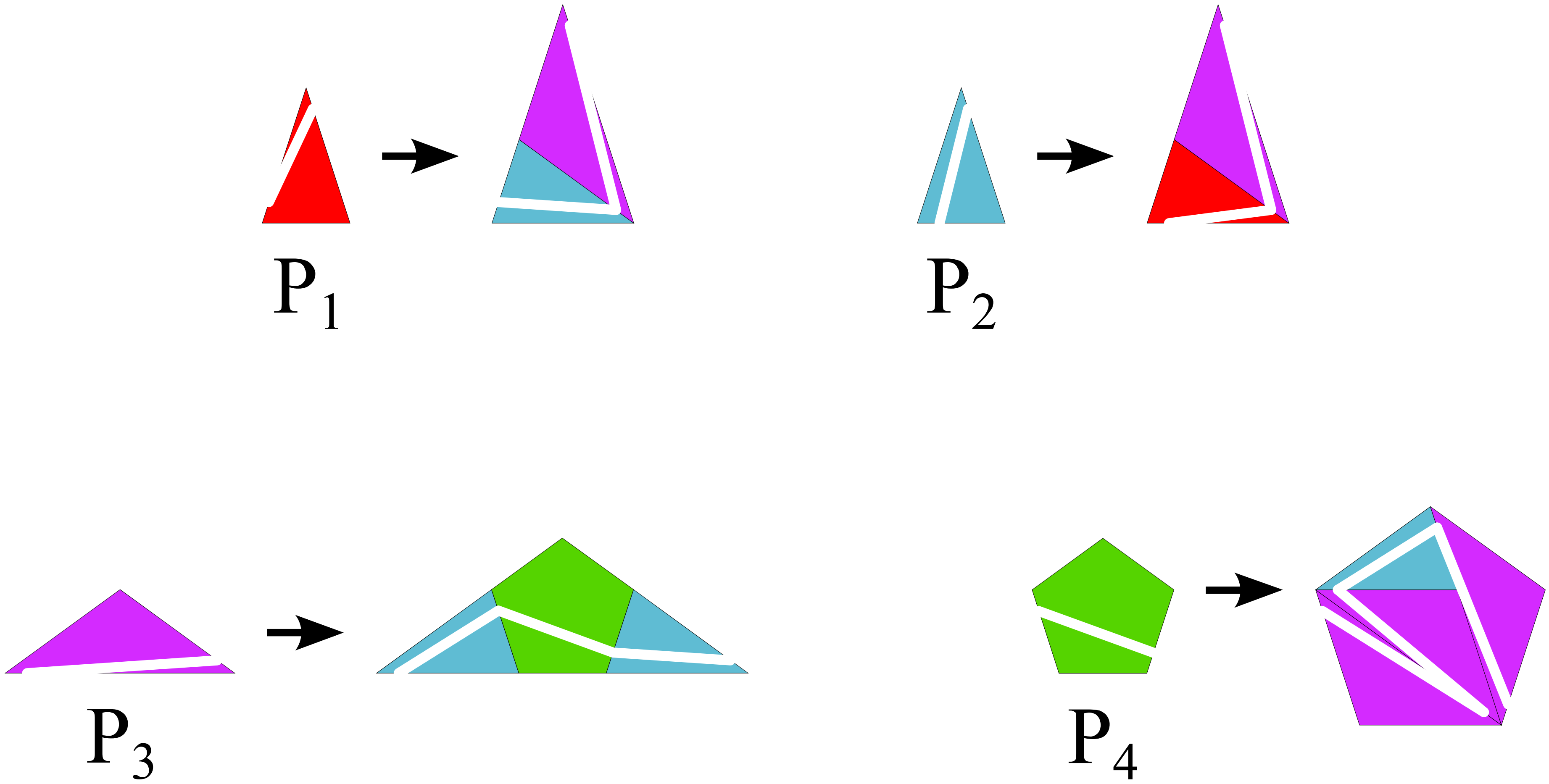
B. Grünbaum and G. C. Shephard. Tilings and Patterns, W. H. Freeman & Co., 1987.

M. Baake and U. Grimm. Aperiodic Order. Vol 1. A Mathematical Invitation, Cambridge University Press, 2013.

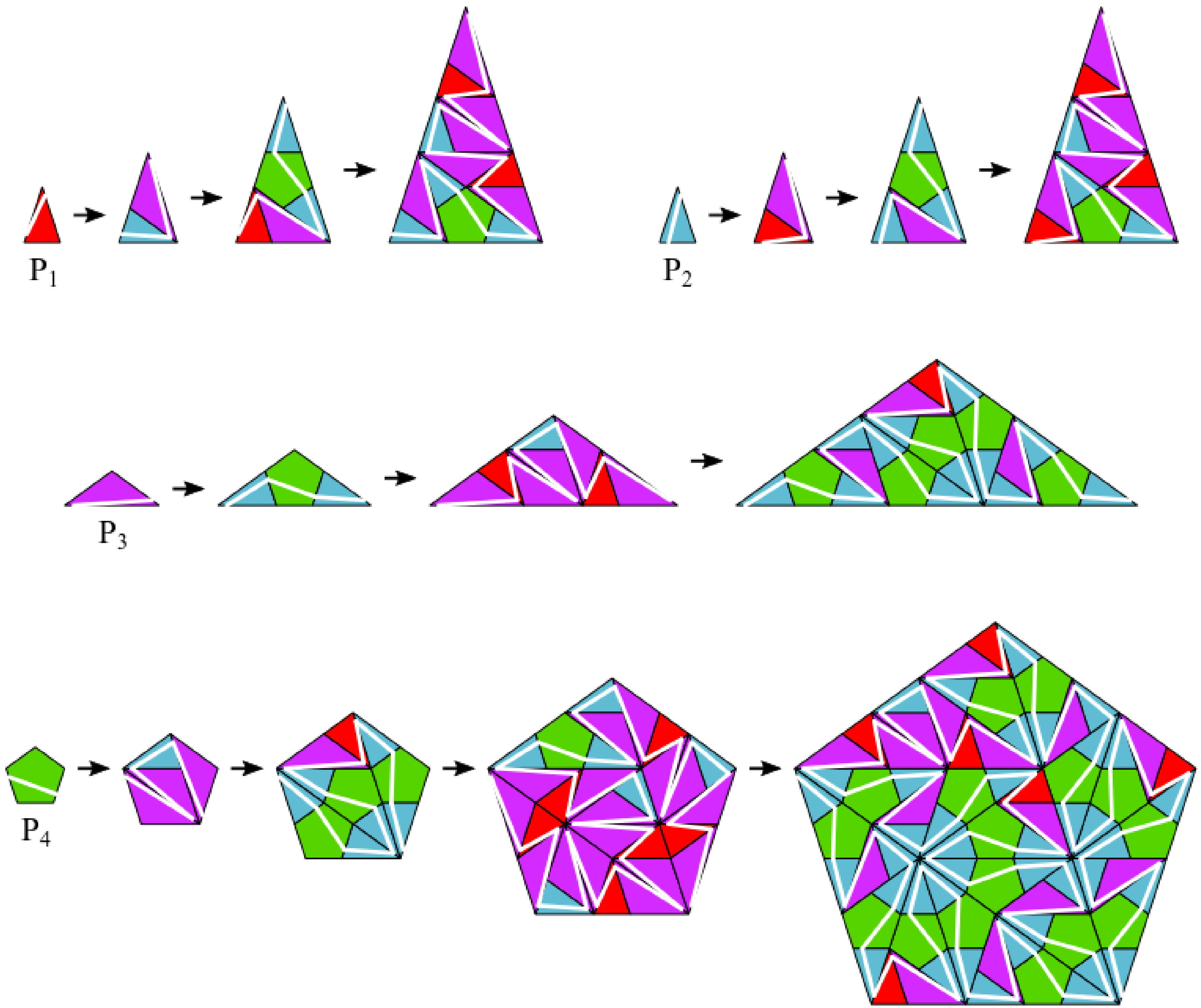
- CAST with vertices supported by the 10-th cyclotomic ring
- 2 proto tiles
- All proto tiles in the shape of isosceles triangles
- All isosceles have unit length
- All inner angles are multiples of $\pi/5$
- Obviously a stone inflation …



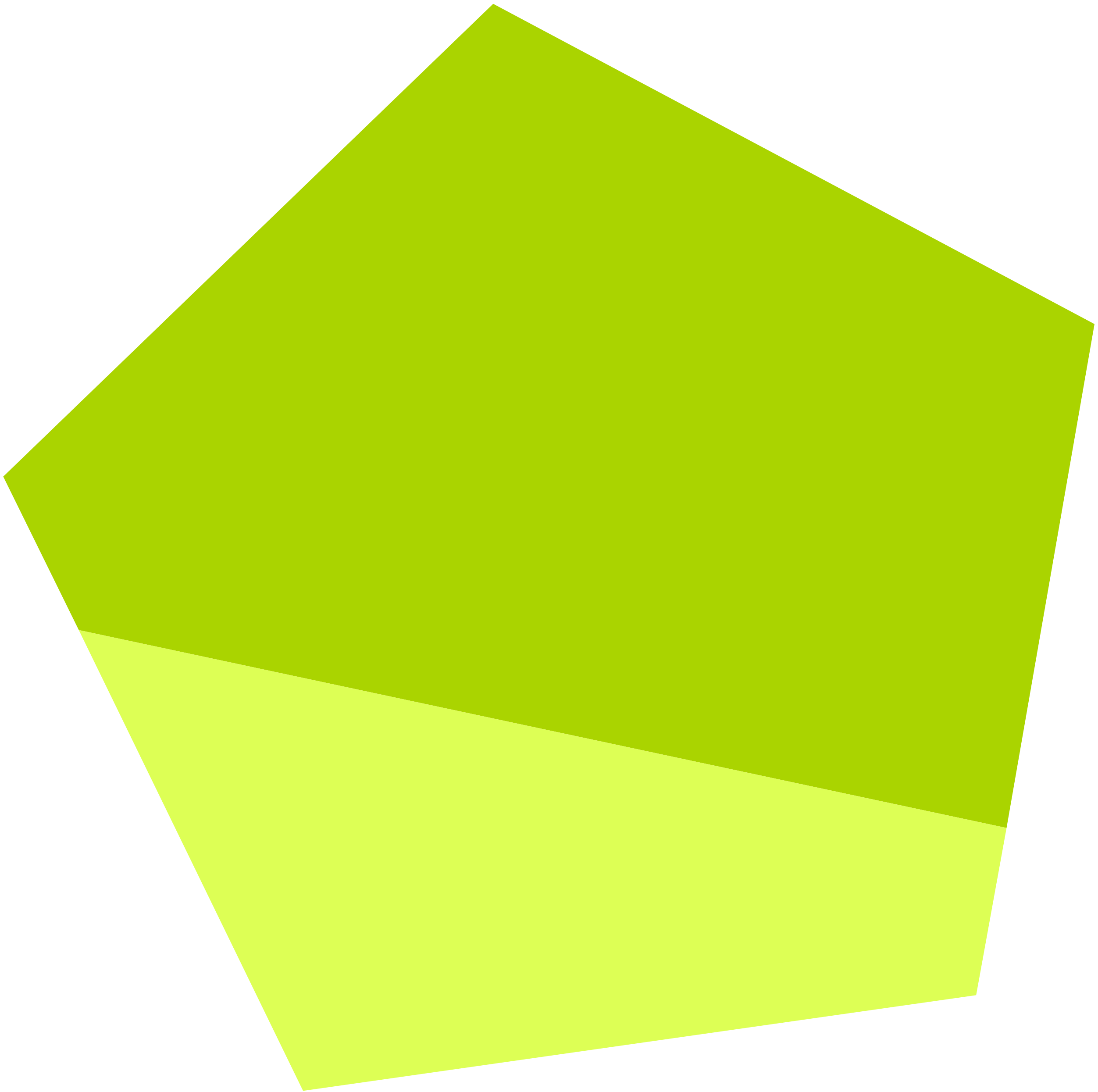
The FASS-Curve of the Regular Pentagon ...



The FASS-Curve of the Regular Pentagon ...



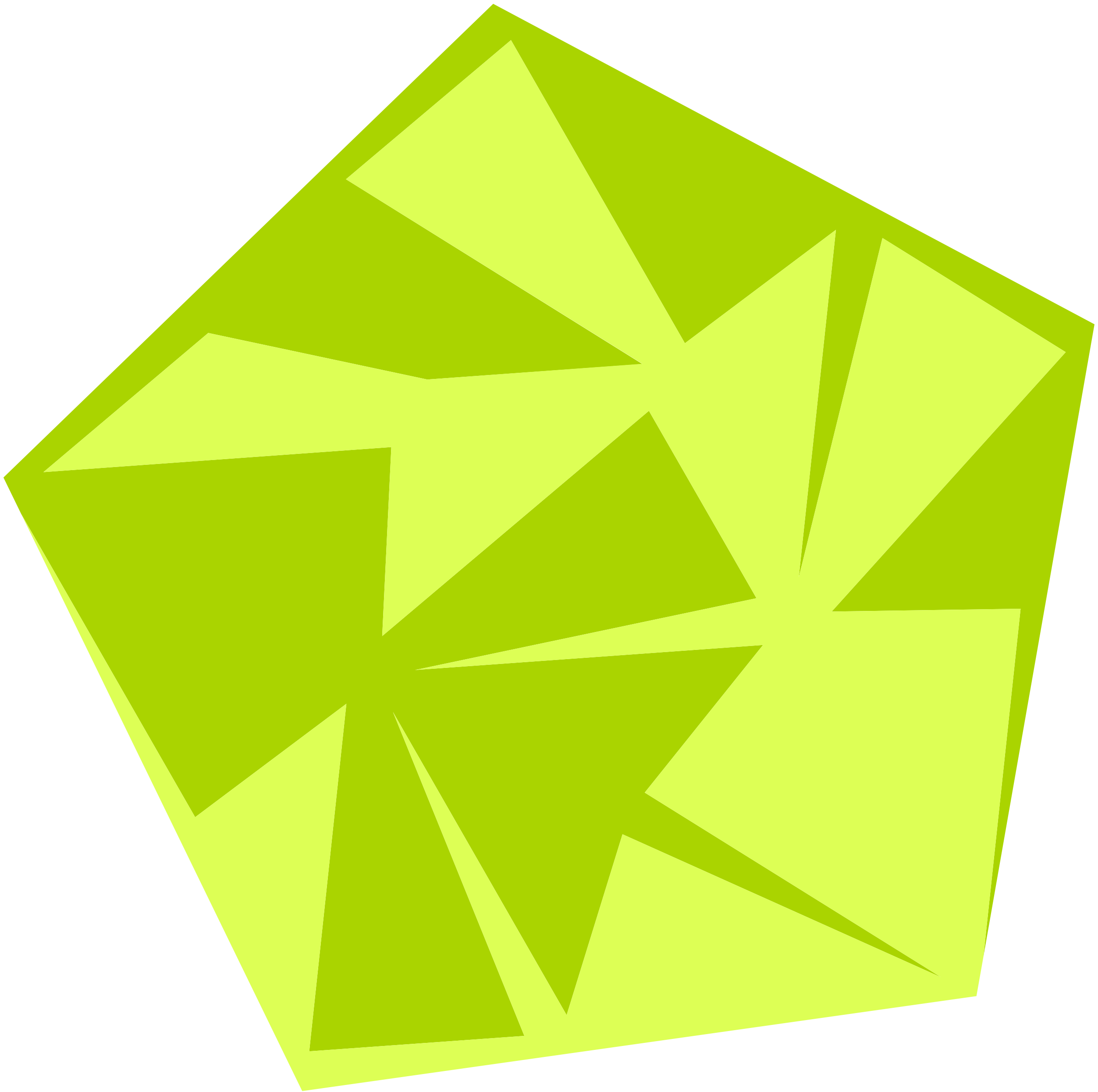
The FASS-Curve of the Regular Pentagon … Initiator



The FASS-Curve of the Regular Pentagon ... 1st Iteration

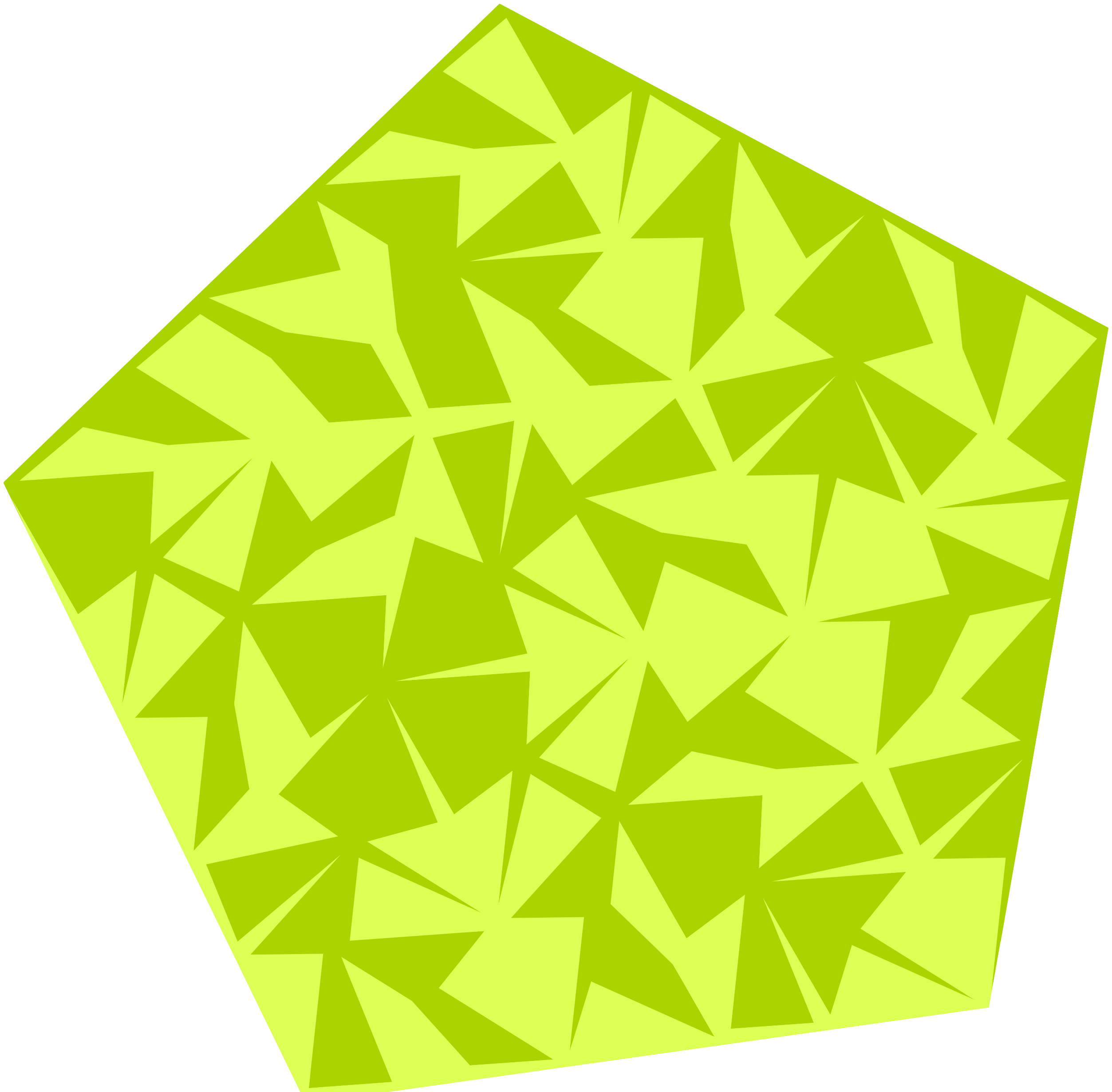
The FASS-Curve of the Regular Pentagon ... 2nd Iteration

The FASS-Curve of the Regular Pentagon … 3rd Iteration

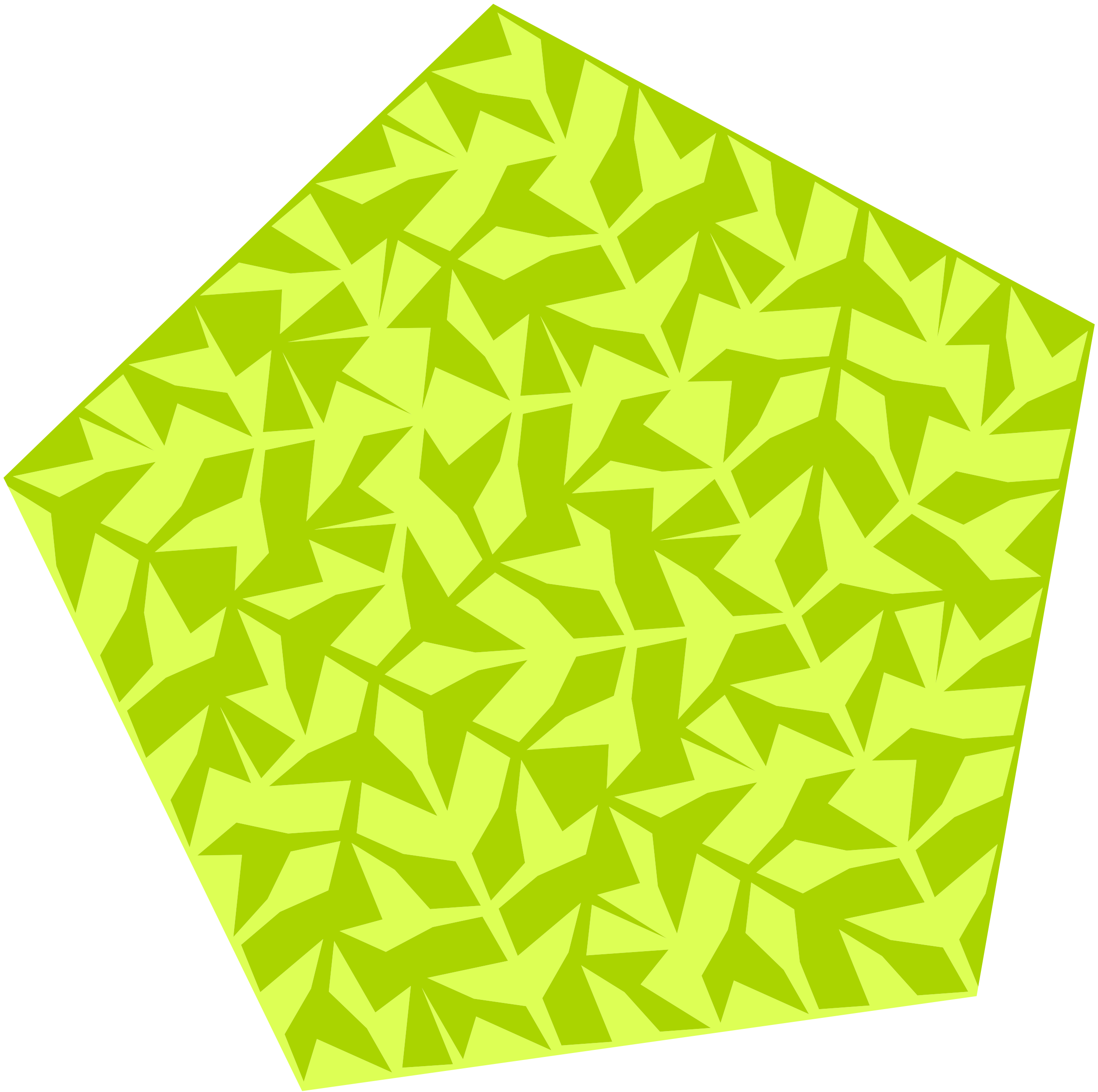


The FASS-Curve of the Regular Pentagon … 4th Iteration

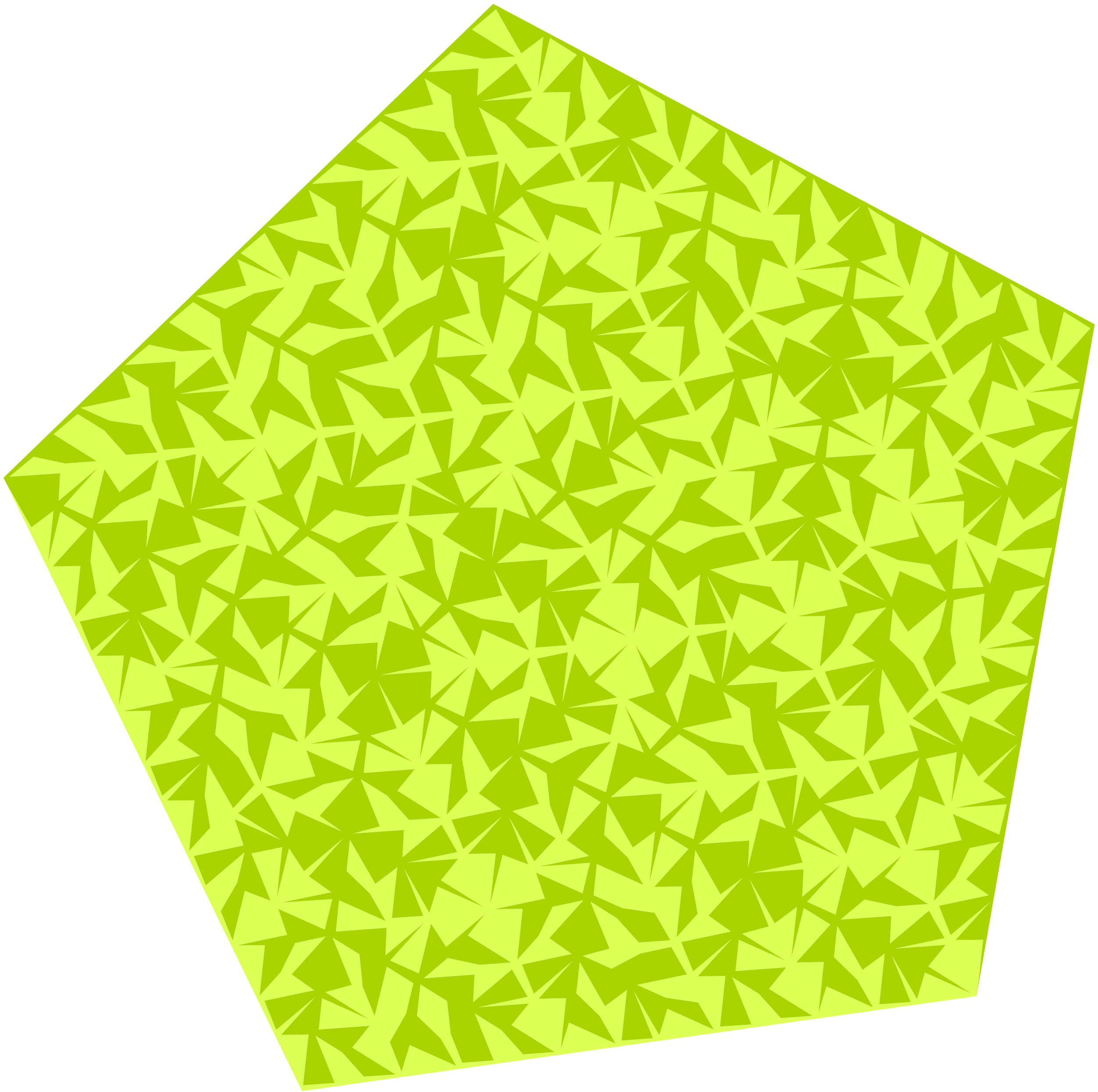
The FASS-Curve of the Regular Pentagon ... 5th Iteration



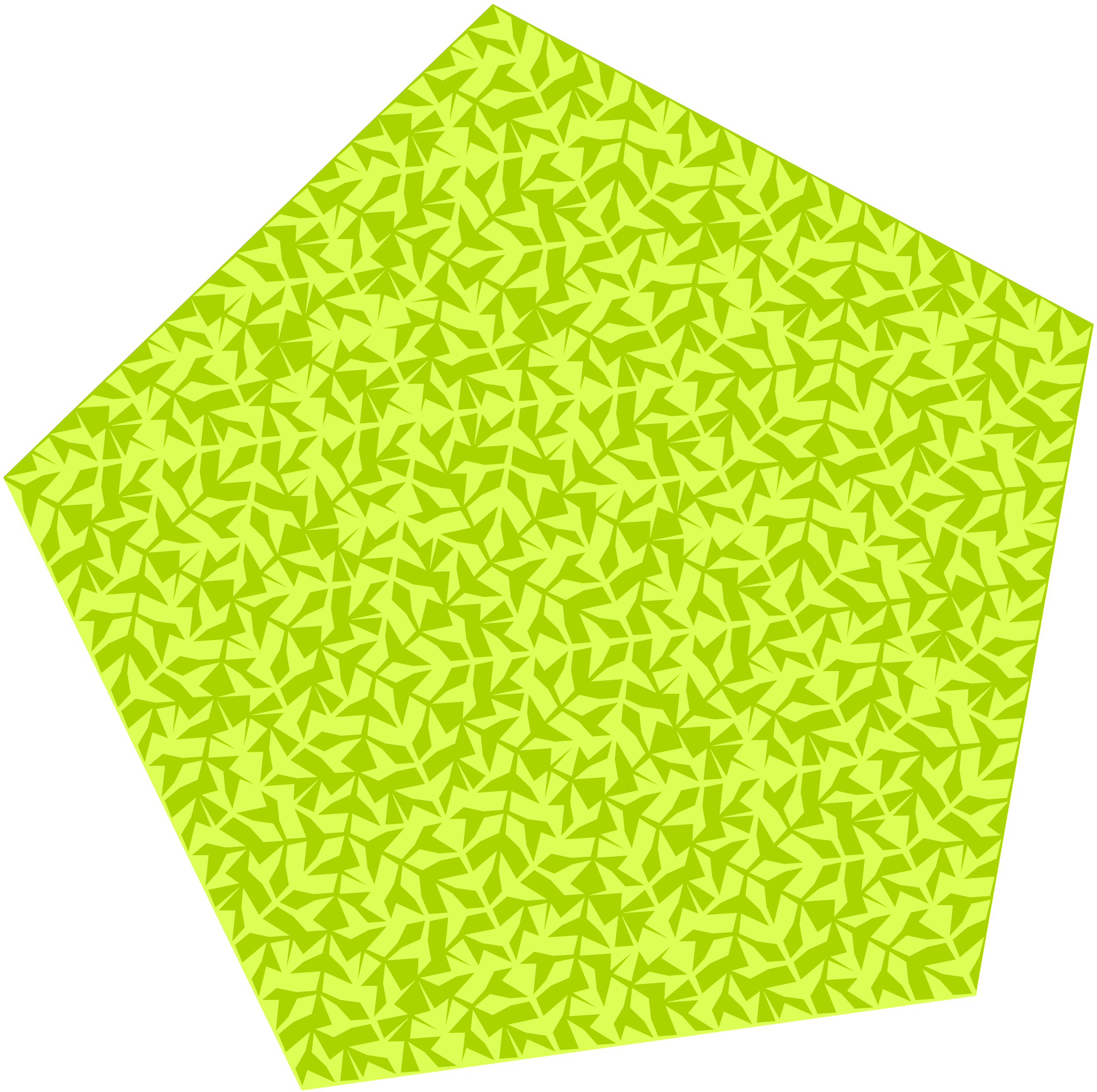
The FASS-Curve of the Regular Pentagon ... 6th Iteration



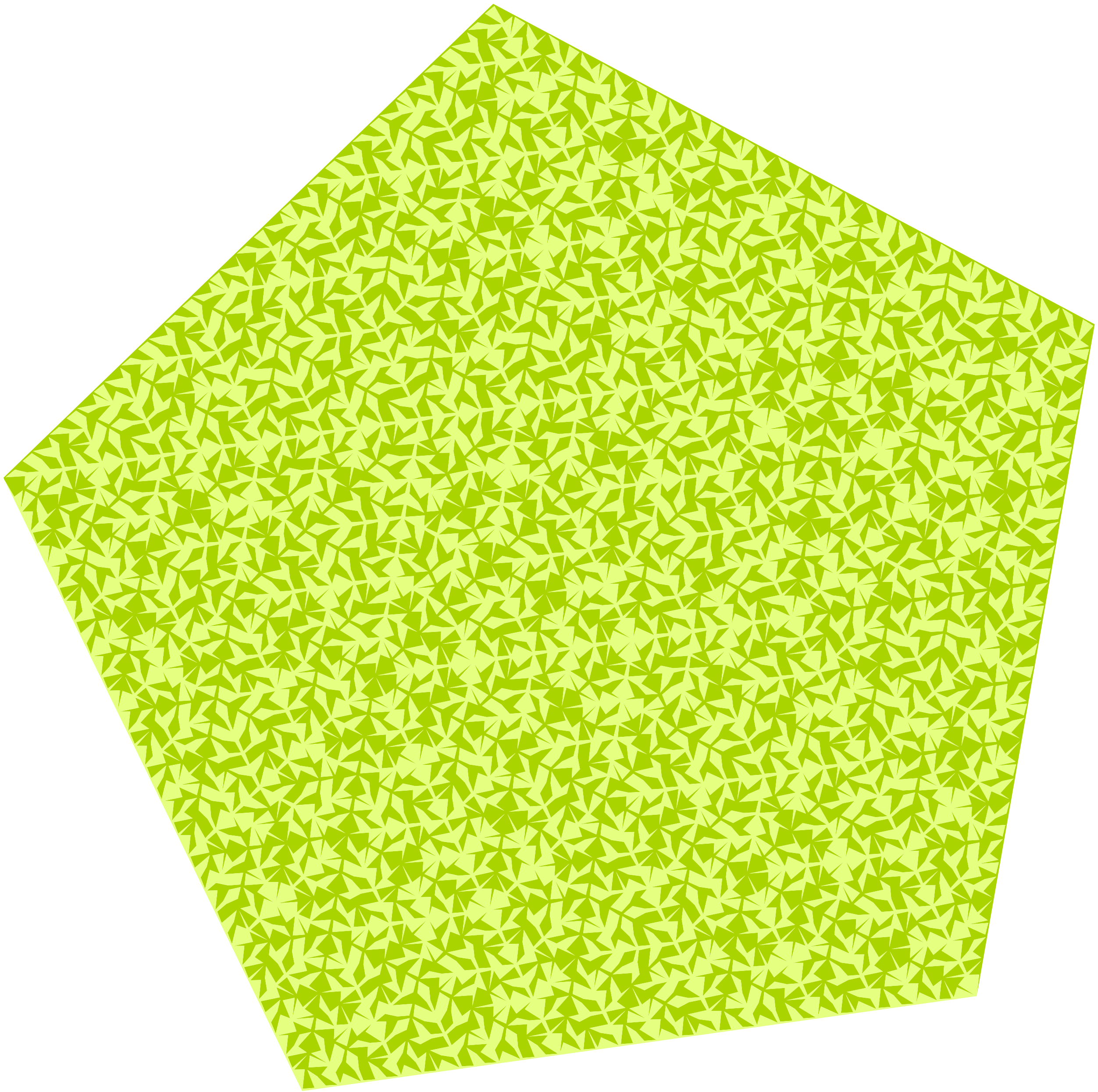
The FASS-Curve of the Regular Pentagon … 7th Iteration



The FASS-Curve of the Regular Pentagon ... 8th Iteration



The FASS-Curve of the Regular Pentagon … 9th Iteration



Thank you !

- **Bridges 2021: Mathematics, Art, Music, Architecture, Culture**
- Dirk Frettlöh (Bielefeld University)
- Christian Mayr (Technische Universität Dresden)
- Klaus-Peter Nischke
- Asta Richter (Technical University of Applied Sciences Wildau)
- Christian Richter (Friedrich Schiller University Jena)
- Jeffrey Ventrella

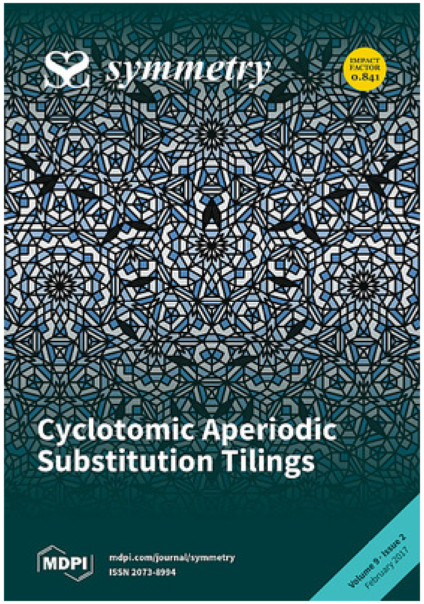
Happy 50th Birthday Richard David James!

Backup / Literature

Cyclotomic Aperiodic Substitution Tilings

Open access article available at MDPI Symmetry

http://www.mdpi.com/2073-8994/9/2/19



Space-Filling, Self-Similar Curves of Regular Pentagons, Heptagons and Other n-Gons

Contribution to Bridges 2021 Virtual Conference

http://archive.bridgesmathart.org/2021/bridges2021-157.html

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